



**NUMBER THEORY
AND RATIONAL NUMBERS**



Number Theory and Rational Numbers

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Academy Introduction

This Academy was designed to provide paraeducators with the skills and knowledge needed to assist students, grades 5 through 8, with mathematics skills taught in the classroom. The course content is designed and adapted from standards recommended by the National Council of Teachers of Mathematics. It includes the specific skill-building areas of number sense; computational techniques for fractions, decimals, and percentages; and their related applications for intermediate and middle-school learners.



***Note to Instructor 1:** This Academy was designed on the premise that the paraeducator is working mathematically at the 5-8 grade level. Activities and timing do not allow for remediation and teaching of basic math concepts. For paraeducators to be successful in this Academy and gain the information at a deeper level, it is **strongly recommended that they complete the Assisting Grades K-4 with Mathematics in the Classroom Academy prior to starting this Academy.** The Assisting Grades K-4 with Math Academy should be seen as a prerequisite unless the paraeducator can demonstrate prior math knowledge.



***Note to Instructor 2:** You will need pattern blocks for this Academy. Resources for pattern block purchase and websites for additional practice are included in the references and product resources section.



Module A

Instructor's Guide



Number Theory and Rational Numbers

Module A: Mathematical Literacy



A. Energizer

Provide an energizer to help participants become familiar with each other and the instructor and to increase the likelihood of greater class participation. If possible, also introduce a sense of anticipation and excitement about working with mathematics.



B. Focus Activity: Math Journal Responses

Paraeducators will participate in an activity that will help them analyze their personal view of mathematics and their perceived knowledge and abilities. This **ungraded** introduction activity takes place prior to the course introduction because it will be biased once participants feel that the correct answers lie in the course goals being presented.



***Note to Instructor:** This is an appropriate time to review the need for well-organized notebooks. Each participant should bring a 3-ring binder for handouts, personal notes and materials used in the class. If this class is offered for credit, reiterate to the participants that they will be taking an **assessment at the end of the class that will be an “open-book test.”** Share the **grading rubric** handout (**GR**) to inform the participants about other requirements that must be met in order for them to receive a grade. Explain that the more highly organized and detailed their personal notebooks are, the more comfortable they will be with the assessment and the more likely they will be to do well on the assessment. It is recommended that the instructor bring a 3-hole punch to class for participants to use and that all handouts are run on 3-hole paper.



B.1 Steps

- Organize the necessary materials. **You will need to provide markers, crayons, colored pencils, glue, scissors, etc., for this activity.**
- Emphasize that this activity will not be graded and will not be viewed or reviewed by other members of the group.
- Use the handout **First Thoughts About Math (H1)**. Provide the handout for each member of the class. Discuss with the class that if they are using a 3-ring binder to organize their materials, they have to include labeled sections in which to keep specific entries.
- Ask participants to respond to the questions on the **First Thoughts About Math** handout. They can respond using writing, drawings or any other format they are comfortable with.

First Thoughts About Math

How would you define mathematics?

What makes mathematics difficult for students?

How would you describe/rate your mathematical ability?

Use transparencies **Response Chart 1** and **2 (T1/T2)** for the first two questions. After partici-



pants have completed the activity (attempt to keep the time for this to approximately 10 minutes), ask them to share their thoughts with the rest of the class. Record their responses on the transparencies.

Do not spend a lot of time discussing or debriefing responses to these questions. They will be revisited in the first goal. Share with the group that their responses are typical of those of others, both students and adults.



C. Lecture: Academy Introduction

Using transparency and handout **Academy Goals (T3/H2)**, introduce and review the contents of the Academy. Review the Academy goals using the following lecture information.

This Academy is designed to provide paraeducators with the skills and knowledge necessary to assist students, grades 5-8, with mathematics skills taught in the classroom. Grades 5-8 are pivotal grades that solidify early elementary concepts and provide a base for high school mathematics courses. Due to the variety of skills developed across grades 5-8, the skills are split over two Academies. This Academy includes the specific skill-building areas of number sense; computational techniques for fractions, decimals, and percentages; and their related applications for intermediate and middle school learners. The course content is designed and adapted from the standards and expectations recommended by the National Council of Teachers of Mathematics.

Two main ideas flow throughout this Academy. First, part of the purpose of this Academy is to build and strengthen paraeducators' mathematics skills. While it is common that paraeducators know some or most of the mathematics in this Academy, it is important that they view the activities at a deeper level of understanding. Explaining concepts to a child requires that the paraeducator view activities and concepts through the eyes of the student. Therefore, paraeducators should reflect on their personal experiences of working with students to explore holes in their understanding and find solutions to helping with weak areas.

The second continuous idea in this Academy is the importance of fully comprehending the activities. The activities were chosen as samples of activities that paraeducators might do with students in their classes. Many contain titles that may feel elementary for this course. Studies show that students retain information better from activities that can be referred to by a name such as "Math Mystery" or "Pack Your Bags" rather than just doing a worksheet. During discussions in the course, refer to the material by referencing the related activity. This helps make connections between concepts.



Number Theory and Rational Numbers

Module A: Mathematical Literacy

The paraeducator will:

1. Identify common misconceptions about mathematics
2. Identify the role of communication in mathematical literacy development
3. Identify the goal of problem solving and its development in the classroom
4. Compare and contrast mathematical literacy and language/reading/writing (literacy) development

Module B: Number Theory and Number Representations

The paraeducator will:

1. Apply number theory concepts to represent numbers in a variety of ways
2. Develop working concepts of factors and multiples
3. Use place-value concepts to represent numbers
4. Develop meanings for common rational and irrational numbers

Module C: Number Sense and Fractional Concepts

The paraeducator will:

1. Use number sense to justify the reasonableness of solutions for a variety of computation and problem-solving situations
2. Use concrete materials to develop fractional concepts for addition and subtraction
3. Use concrete materials to develop fractional concepts of multiplication and division

Module D: Decimal and Percentage Concepts

The paraeducator will:

1. Use concrete materials to develop decimal concepts
2. Use concrete materials to develop percentage concepts
3. Connect decimal and percentage concepts

Module E: Making Connections

The paraeducator will:

1. Develop conversion algorithms among fractions, decimals, and percentages
2. Compare rational and irrational numbers through equivalent forms, using a variety of strategies
3. Apply computational procedures for fractions, decimals and percentages to real-world problems



*** Note to Instructor:** If several participants in the course have already completed the Assisting Grades K-4 with Mathematics in the Classroom Academy, you can decrease the time spent on portions of Module A, as some of the material is identical to the K-4 Academy. See below for suggested alterations.

Alterations for Module A Goals:

Goal 1: Complete as written or shorten **First Thoughts** ...

Goal 2: Complete as written

Goal 3: Complete as written

Goal 4: Skip



Goal 1: Identify common misconceptions about mathematics.



1.1 Discussion: Common Misconceptions About Mathematics

The paraeducator will define misconceptions about “mathematics” that they encounter while working with students and their parents.



1.1.1 Steps

- Direct participants to return to the **First Thoughts** handout (**H1**) while you return to the transparencies **Response Chart 1** and **2 (T1/T2)**. Briefly review the answers/responses. Ask participants to review their private responses. Compare their responses to those listed in the next step.
- Use the **Question 1: Common Responses** transparency (**T4**). Discuss common answers and misconceptions about mathematics from both children and adults. Many other answers are possible; these are only a few that will help spark discussion:

Question 1: Common Responses

- ▲ Word problems (problem solving)
 - ▲ Numbers
 - ▲ Rules
 - ▲ Problems
 - ▲ Memorization
 - ▲ Skills
 - ▲ Drill
 - ▲ Homework
 - ▲ Not fun
 - ▲ Too hard
- Use the **Question 2: Common Responses** transparency (**T5**). Discuss the common answers listed. This is a particularly important question for those working with middle- and secondary-grades students, as this is the time when they often struggle most. Many other answers are possible; these are only a few that will help spark discussion:

Question 2: Common Responses

- ▲ Didn't get the “basics”
 - ▲ Vocabulary
 - ▲ Too many processes
 - ▲ “Word problems”
 - ▲ Not interested
- Use the **Response Chart 3** transparency (**T6**). Brainstorm with attendees why their personal perceptions might affect their students. Record their answers. Possible responses to look for include:



- ▲ Only certain people can do math
- ▲ Not successful
- ▲ Too much vocabulary (language problems)
- ▲ Don't see math in their everyday life like other literacy skills
- ▲ Socio-economic-cultural background
- ▲ Gender differences (*Note:* Current studies show that girls are turned off to mathematics by fourth grade)



1.2 Lecture: Defining Mathematics

Return to the **Question 1: Common Responses** transparency (**T4**). Ask attendees which responses on the list represent arithmetic. Check off the responses that represent arithmetic (all but the last two). Discuss with class participants that common misconceptions are often the result of confusing definitions. Use the transparency **Arithmetic** (**T7**).

Arithmetic

Arithmetic – calculations involving predefined rules

Arithmetic is an important part of mathematics but it does not encompass all that is included in defining mathematics. Arithmetic skills such as adding, subtracting, using fractions, algebra, etc., are all markedly the reason why students/adults say they dislike *mathematics*.



***Note to Instructor:** Remind participants that many of the students whom paraeducators work with may have low arithmetic skills, but not necessarily low mathematics skills.

Explain that this statement may have caused some confusion. Another way to look at it or to examine the statement more closely would be to ask for a show of hands for the following:

“How many of you enjoy the following?”

- Doing puzzles
- Art/drawing
- Landscaping
- Cooking
- Decorating

Explain that all of these activities are mathematics activities. They all involve skills that mathematicians use. In the following modules we will be examining this much more closely. Use the transparency/handout **Mathematics is . . . (T8/H3)**. As the points of the transparency are reviewed, explain where else they will be covered in the Academy. Discuss what Reys, Suydam, and Lindquist (1992) say about how to define mathematics.



Mathematics is ...

Mathematics is ...

1. a study of patterns and relationships (all modules)
2. a way of thinking (all modules)
3. an art (involves creativity – not just rules)
4. a language (we will spend more time on this one)
5. a tool (used in almost everything we do)

Reys, Suydam, and Lindquist (1992)

Remind participants that in order to understand mathematics as more than arithmetic, they must recognize that math exists in almost every aspect of their lives, starting when they get out of bed each morning (telling time on a clock) and ending late at night when they choose their clothes for the next day (probability).



Goal 2: Identify the role of communication in mathematical literacy development.



2.1 Lecture: Math as a Language

The paraeducator will learn that part of understanding the definition of mathematics includes seeing math as a language.

Students have oral language before they can read. Those skills are developed from early in life. Their language skills help them to develop further literacy skills, which include word recognition, reading, and comprehension. They also grow up seeing and hearing people around them using communication, reading, and writing skills. They are *obviously and observably* learning language and how to communicate, therefore, further extending their reading and writing skills.

Learning mathematics is not an obvious process like language. Young students do not have observable early experiences with mathematics; they, and the adults raising them, often do not recognize mathematics in use around them. Students in the middle grades have even more difficulty recognizing the value and existence of the abstract mathematics they begin to learn. When mathematics, commonly arithmetic, is introduced, it often appears and is practiced as something outside of their regular experiences. For students for whom English is not their first language, mathematics appears to be another foreign concept to be learned.

Mathematics functions along similar lines to language. It has its own rules, terms, and symbols that require the same practice as learning a language. Treating math as a language changes the methods by which it is taught.



Using the transparency **Learning a Second Language (T9)**, ask attendees to share responses regarding what they believe is required to learn a second language. Record their responses on the transparency. Possible responses include:

- Exposure
- Practice/use
- Taking classes
- Reading and writing it
- Immersion in the culture representing the language

Now make the connection between learning a foreign language and learning mathematics. There are many similarities. Use the **Math as a Language** handout and transparency (**H4/T10**) to help explain math as a language.

Math as a Language

To learn the math language students/learners must:

1. Read it
2. Write it
3. Speak it
4. Do it

Read it – Implies more than just reading directions. Reading involves recognizing vocabulary, forming and interpreting math sentences and following other students' written ideas (e.g., $3 + 2 = 5$ as a sentence could be “putting together 3 cars and 2 more cars is [gives] how many?; gives me 5 cars”).

Write it – Implies more than homework problems. This means explaining the problem with drawings, words, labels and vocabulary. This reinforces the reading skill.

Speak it – Teachers should not be the only ones speaking the language. Students need practice with the language. They need to hear themselves, hear models (from teachers or other adults) and hear their peers. This requires that students discuss math or provide input in mathematical discussions. This includes having children talk out their work before ever putting pencil to paper.

Do it – This requires individual practice. It means trying out the language skills in homework practice, in group conversations or in a reflective journal. Putting all of the prior skills together allows the student to use the language.

The four previous ideas include many literacy skills: reading, writing, communicating, listening, and speaking. These ideas are not traditionally thought of as mathematics skills. If we link language skills and mathematics, we must be open to changing practices and beliefs to help students find more success than just the traditional methods that are typically offered. *We are talking about a change in philosophy.* It is important to recognize the elements of this philosophy and why they are important.



Discuss the importance of this change of philosophy for mathematics.

- Students can link mathematics with successful prior learning.
- Students can build on language skills they may already possess and add to their language base with the language of mathematics.
- Students can better communicate their understandings with both the teacher and their peers.
- Students can begin to see the mathematics process as a work in progress when they hear and participate in the process.

This change in philosophy emphasizes the role of communication in mathematical literacy development. Language is an important element in literacy. The purpose of language is to develop clear communication of ideas and processes. This is as important in mathematics as it is in everyday social communication.



2.2 Activity: Creating a Math Journal

The paraeducator will continue creating his/her math journal as a model for use with students. This portion of the journal will focus and build on the need for math communication.



2.2.1 Steps

- Using the transparency **Math Journal (T11)** and the **Reflections, Problem Solving** and **Reference** handouts (**H5/6/7**), introduce the math journal.
- The handouts may be used as dividers for each section as reminders.
- Explain that this journal will consist of three sections. Each participant is expected to provide his or her own paper for taking notes within each section. Handouts should be filed appropriately within the following organization:
 - ▲ reflection
 - ▲ problem solving
 - ▲ reference
- Discuss the first section, Reflections, including the following information about how to use the journaling process. (The **First Thoughts About Math** handout [**H1**] should be filed under this heading.)
 - ▲ Reflections from students often provide new insight into student understanding, often more so than homework and performance of math problems.
 - ▲ Journals help students practice communication for mathematics.
- The second section is for recording in-class problem-solving work.
 - ▲ This keeps student work together in one place and makes it easy to go back and look at strategies and processes. When paraeducators begin carrying over some of the strategies and skills they are learning with students they will frequently refer to this section. Being organized provides easy access to the materials and increases the likelihood that the paraeducators will make good use of their learning.
- The last section of the journal is used as a reference tool.
 - ▲ Each working period, students should be asked to put unfamiliar or new concepts or vocabulary in their reference book.



- ▲ Help should be given with the proper spelling of the vocabulary words, but students should define the word/concept in their own words and use pictures where necessary.
- ▲ Details should also be clarified to make sure students' definitions are mathematically sound.
- The journal is designed to be something students will use on their own, and should be individually organized for best personal use.
- The journal supports paraeducators when no teacher is present and helps them be successful.
- Use of a math journal is an important practice when working with students who are second-language learners; math may seem difficult and overwhelming because of the multiple unique terms used. The paraeducator can keep a list of new learning for each student and be better prepared to seek assistance with problem teaching areas.
- A journal also provides the opportunity for the necessary repetition of ideas.



Goal 3: Identify the goal of problem solving and its development in the classroom.



3.1 Lecture: Define Problem Solving

Use the transparency and handout **Problem Solving: A Definition (T12/H8)**.

The National Council for Teachers of Mathematics (NCTM) (2000) defines problem solving as follows:

Problem Solving: A Definition

Problem solving means engaging in a task for which the solution method is not known in advance.

In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings.

Solving problems is not only a goal of learning mathematics, but also a major **means** of doing so.

Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking.

(see <http://standards.nctm.org/document/chapter3/prob.htm>)



While this is a lengthy definition, it includes very important points about problem solving:

- Problem solving should not be a separate unit of learning.
- Problem solving should also not be confused with “word problems.” Many word problems are merely a rephrasing of an arithmetic problem.
- Good application problems allow for problem solving beyond the use of a simple algorithm (rule).
- Problem solving is the adhesive that holds the math curriculum together.

Young children are by nature curious and are typically very flexible when they encounter new learning. Most often they do not fear new challenges and typically like the excitement of solving a problem or challenge. They enjoy experimenting with new methods of problem solving and do not fear mistakes or failure. The need to solve a proposed problem generates the need for acquisition of the skills. *Problems must be posed where solutions are not immediately obvious.* This encourages children to take a risk and strive to find a solution.

Middle grade students present a challenge as they often come with more negativity towards “word problems” due to poor experiences with problem-solving activities. They are less likely to experiment with solving strategies and more likely to stop the process when a basic failure arises in the solution attempt.



3.2 Activity: Problem-Solving Practice

The paraeducator will participate in an activity requiring practice of personal problem solving-skills and examination of the problem solving skills of others.

Materials:

- Handout **Centimeter Grid Paper (H9)**
- Transparency **Problem Solving (T13)**



3.2.1 Steps

- Divide the class into groups of 4-6.
- Provide each person with the handout **Centimeter Grid Paper**.
- Using the **Problem Solving** transparency, pose the following question to the group:

Problem Solving:

“Place the numbers 1-9 in a 3x3 grid so that each row, column and diagonal has the sum of 15. No number may be used more than once.”

- Direct the groups to work through this problem. Ask someone in each group to take notes to be shared with the class. Their notes should reflect how the group came to answers or conclusions; what was their thinking?
- Share group answers with the entire class.



3.3 Lecture: Problem Solving and the Magic Square Activity

The following magic square activity uses only the operation of addition, but presents many problem-solving tasks. While it would seem that there is only a single process to solve this problem, groups will be surprised to see how each person attacks the problem differently. Groups may need to clarify *row*, *column*, and *diagonal*. Some groups will find all possible sums of 15, whereas others will simply use trial and error. A child-friendly approach is to put all nine numbers on separate squares, cut them out and physically manipulate the pattern. Allowing students to work uninterrupted and choose their own problem-solving strategy provides opportunities for some creativity and encourages students to work the problem at their own level of understanding.

Encourage groups during their discussions to record any rules that they created and patterns they discovered while solving this problem. For example, numbers 8 and 9 could not be in the same row because the sum is too large. Rules may be recorded on the **Problem Solving** transparency (T13) of the problem as they arise to assist other groups. These should be recorded in the journal.

When groups finish, have them share their answers and further question if there is more than one possible solution. Allowing students to generate their own questions or problem variations also provides opportunities to make problem solving personal. The multiple solutions that will arise are reflections of each other's thinking.

Possible Magic Square Solutions

(others are possible):

6	1	8
7	5	3
2	9	4

8	1	6
3	5	7
4	9	2

4	9	2
3	5	7
8	1	6

As a group, discuss what math skills/concepts are necessary to solve the problem. After soliciting responses from the group use the **Skills and Concepts** transparency (T14) to review or to provide information.

Skills and Concepts (Suggestions)

- Addition rules
- Recognizing that the order does not matter when adding numbers; the answer is the same (commutative property of addition)
- Pattern development
- Organizing data

It is important for paraeducators to understand that as students tackle new problems, it is



very helpful if they have a plan. This is an area where paraeducators can be extremely helpful in the classroom. George Polya (1957) produced a quick guide to help students develop into good problem-solvers. Use the transparency and handout **Problem Solving with a Plan (T15/H10)**.

Problem Solving with a Plan

Steps for Problem Solving:

1. Understand the problem
2. Devise a plan
3. Carry out the plan
4. Look back

Understand the problem

- Rephrase the problem in your own words
- What are you trying to find or do?
- Unknowns?

Devise a plan

Throughout any math course, students need tools. Tools are not just rules. Tools involve strategies. While names are presented here, the names are not standard nor should students focus on memorizing the titles. They are simply ways to refer to processes that can be useful in solving problems. This is only a partial list for discussion:

Problem-Solving Strategies

- Look for a pattern
- Create a simpler problem
- Make a table
- Draw a diagram
- Write an equation
- Guess and check
- Work backward
- Identify a subgoal for the problem (involves breaking a problem down)

As students develop a plan, they often want to get to work right away rather than finish their plan. For young children, talking out a plan and recording it is often a good place to start developing problem-solving skills.

Carry out the plan

- Implement the strategy
- Perform any necessary computations
- Keep a record of your work

Look back

- Check results
- Revise the plan
- Make connections to other problems
- Try again if necessary



This is often the most difficult step for students. They assume that when they get to the end, it is over. Many students get frustrated if the problem is incorrect the first time. Part of problem solving is learning to revise and try again. This supports the old adage, “If at first you don’t succeed, try, try again.”

This method also very closely follows Bloom’s taxonomy. Learners go through the stages of knowledge and comprehension as they come to understand the problem; then they use application skills as they devise a plan to solve the problem. Next, they carry out the plan and begin to analyze the plan for success. Lastly, they look back at their activity and synthesize and evaluate their learning.



Goal 4: Compare and contrast mathematical literacy and language/reading/writing (literacy) development.



4.1 Activity: Compare and Contrast

The paraeducator will compare and contrast mathematical literacy and language/reading (literacy) development.

Materials:

- **Comparing and Contrasting Math and Reading Literacy transparency (T16)**
- **Comparing and Contrasting Math and Reading Literacy handout (H11)**



4.1.1 Steps

- Using the transparency and handout **Comparing and Contrasting Math and Reading Literacy (T16/H11)**, direct paraeducators to work in small groups.
- Ask them to respond to the question: “What skills/expectations do you think a student has to develop in order to have the appropriate language/reading/writing literacy skills needed in grades 5-8?”
- Direct them to assign a note taker and representative from each group to represent their responses to the class during the large-group followup portion of this activity.
- After groups have had time to meet and share, ask them to share their responses with the entire class. Record their responses on the transparency. Share the group ideas to be recorded on a class list for literacy. If the recorded responses do not include the items listed below, add them to the list.

Comparing and Contrasting Math and Reading Literacy

Reading

- ▲ Letter recognition
- ▲ Letter sounds
- ▲ Word recognition
- ▲ Word comprehension
- ▲ Sentence production (rules)



- ▲ Sentence comprehension
- ▲ Paragraph summarization
- ▲ Order of events
- ▲ Prediction

Language

- ▲ Letter sounds
- ▲ Appropriate pronunciation
- ▲ Ability to communicate ideas
- ▲ Ability to formulate questions and responses

Writing

- ▲ Letter formation
- ▲ Sentence production
- ▲ Ability to communicate ideas

- Use the transparency **Defining Literacy (T17)**.

Defining Literacy

Literacy	Mathematical Literacy
Fundamentals (letters, sounds, etc.)	Fundamentals (numbers, symbols, etc.)
Rules (punctuation, spelling, etc.)	Rules (algorithms, order of operations, etc.)
Sentence production to represent ideas	Mathematical sentences (equations)
Practical uses outside the classroom (magazines, signs, conversations, etc.)	Practical uses outside the classroom (money, time, distance, shapes, etc.)
Comprehension	Ability to explain a problem or process
Prediction	Patterns and problem solving
Communication	Communication

- Distribute copies of the handout **Defining Literacy (H12)** and ask the small groups to reconvene and discuss how the skills they listed previously fall into the categories listed in the handout: fundamentals, rules, sentence production, practical uses, comprehension, prediction and communication.
- Direct the groups to discuss the categories as they apply to mathematical literacy.



4.2 Lecture/Conclusion: Defining Mathematical Literacy

The term *mathematical literacy* implies that mathematics is not a static, one-sided concept. This is the new philosophy promoted in many current mathematics programs starting in the early elementary grades.



As reflected on the handout **Defining Literacy (H12)**, success in mathematics requires the same expectations as for reading, writing, and language skills.

While individual skills may not line up exactly, foundational concepts tend to run parallel with a new definition of mathematics. Paraeducators may need help in discussing and reorganizing their thinking. This concept will be reinforced and revisited throughout the course.



Module A

Handouts

Please respond to the following journal questions. Feel free to respond using writing, drawings or any other format with which you are comfortable.

1. How would you define mathematics?
2. What makes mathematics difficult for students?
3. How would you describe/rate your own mathematical ability?



Overview of Mathematics in the Classroom

Number Theory and Rational Numbers

Module A: Mathematical Literacy

The paraeducator will:

1. Identify common misconceptions about mathematics
2. Identify the role of communication in mathematical literacy development
3. Identify the goal of problem solving and its development in the classroom
4. Compare and contrast mathematical literacy and language/reading/writing (literacy) development

Module B: Number Theory and Number Representations

The paraeducator will:

1. Apply number theory concepts to represent numbers in a variety of ways
2. Develop working concepts of factors and multiples
3. Use place-value concepts to represent numbers
4. Develop meanings for common rational and irrational numbers

Module C: Number Sense and Fractional Concepts

The paraeducator will:

1. Use number sense to justify the reasonableness of solutions for a variety of computation and problem-solving situations
2. Use concrete materials to develop fractional concepts for addition and subtraction
3. Use concrete materials to develop fractional concepts of multiplication and division

Module D: Decimal and Percentage Concepts

The paraeducator will:

1. Use concrete materials to develop decimal concepts
2. Use concrete materials to develop percentage concepts
3. Connect decimal and percentage concepts

Module E: Making Connections

The paraeducator will:

1. Develop conversion algorithms among fractions, decimals, and percentages
2. Compare rational and irrational numbers through equivalent forms using a variety of strategies
3. Apply computational procedures for fractions, decimals and percentages to real-world problems



Mathematics is ...

1. a study of patterns and relationships
2. a way of thinking
3. an art
4. a language
5. a tool

-Reys, Suydam, and Lindquist (1992)



Math as a Language

To learn the math language students/learners must:

1. Read it
2. Write it
3. Speak it
4. Do it

Read it – Implies more than just reading directions. Reading involves recognizing vocabulary, forming and interpreting math sentences and following other students' written ideas (e.g., $3 + 2 = 5$). As a sentence this could be “putting together 3 cars and 2 more cars is [gives] how many?; gives me 5 cars”).

Write it – Implies more than homework problems. This means explaining the problem with drawings, words, labels and vocabulary. This reinforces reading skills and provides a way to practice.

Speak it – Teachers should not be the only ones speaking the language. Students need practice with the language. They need to hear themselves, hear models (from teachers or other adults) and hear their peers. This requires students to discuss math or provide input during mathematical discussions. This includes having children “talk out” their work before ever putting pencil to paper.

Do it – This requires individual practice. This means trying out the language skills in homework practice, in group conversations or in a reflective journal. Putting all of the prior skills together allows the student to use the language.



Reflections

- My personal responses and insights into student understanding
- Thoughts and insights about my own learning
- “AH-HA” moments; the light bulb just went on for me, and I want to remember what I learned and how I learned it
- Writing about the processes I am learning will help me communicate my learning to others



Problem Solving

- Helps me remember how I solved a problem when I can review my own process
- Keeps my work samples organized and enables me to find and reference them easily
- I will be able to review my own work to help me clarify my own process when I am working with students



Reference

- I can record unfamiliar concepts
- I can record definitions and other language that will be helpful later
- I can keep a list of new learning for each student
- I can be better prepared to seek assistance with problem teaching areas



Problem Solving: A Definition

Problem solving means engaging in a task for which the solution method is not known in advance.

In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings.

Solving problems is not only a goal of learning mathematics, it is also a major means of doing so.

Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking.

Important points about problem solving:

Problem solving should not be a separate unit of learning.

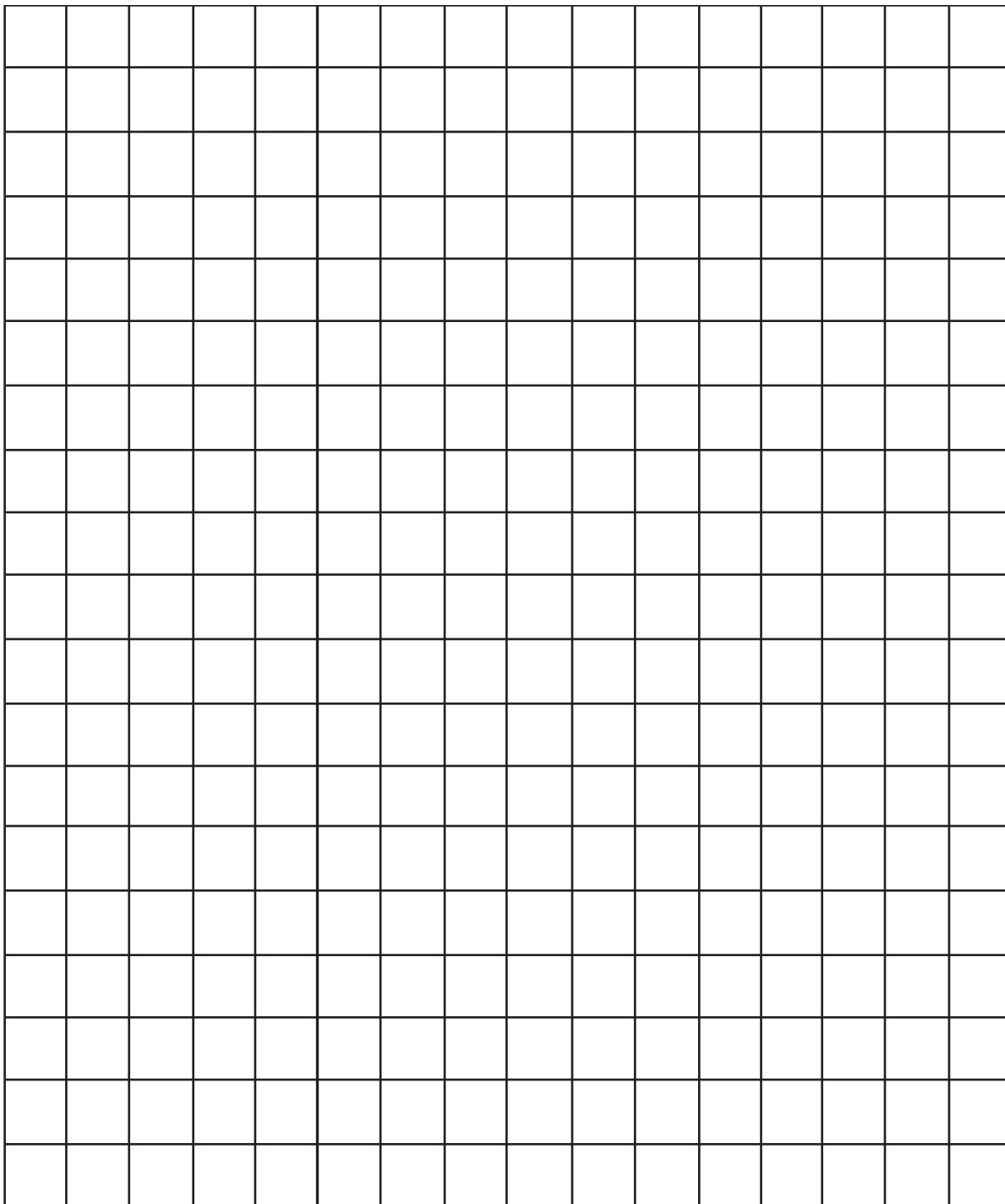
Problem solving should not be confused with “word problems.” Many word problems are merely a rephrasing of an arithmetic problem.

Good application problems allow for problem solving beyond the use of a simple algorithm (rule).

Problem solving is the adhesive that holds the math curriculum together.



Centimeter Grid Paper





Problem Solving with a Plan

Steps for Problem Solving:

1. Understand the problem
2. Devise a plan
3. Carry out the plan
4. Looking back

Understand the problem

- Rephrase the problem in your own words
- What are you trying to find or do?
- Unknowns?

Devise a plan

Students need tools. Tools involve strategies. The following is a list of problem-solving tools or strategies that students can use. The strategies are simply ways to refer to processes that can be useful in solving problems. This is only a partial list, and many other strategies may be used.

Problem-Solving Strategies

- Look for patterns
- Create a simpler problem
- Make tables
- Draw diagrams
- Write equations
- Guess and check
- Work backward
- Identify a subgoal for problems (this involves breaking a problem down)

Carry out the plan

- Implement the strategy
- Perform any necessary computations
- Keep a record of your work

Look back

- Check results
- Revise the plan
- Make connections to other problems
- Try again if necessary



Comparing and Contrasting Math and Reading Literacy

Instructions: Please record personal responses, small-group responses and responses from whole-class discussions in the columns below.

Reading

Language

Writing



Defining Literacy

Literacy	Mathematical Literacy
Fundamentals (letters, sounds, etc.)	Fundamentals (numbers, symbols, etc.)
Rules (punctuation, spelling, etc.)	Rules (algorithms, order of operations, etc.)
Sentence production to represent ideas	Mathematical sentences (equations)
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Comprehension	Ability to explain a problem or process
Prediction	Patterns and problem solving
Communication	Communication



Module A

Transparencies



Response Chart 1

Define Mathematics

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Response Chart 2

What Makes Mathematics Difficult for Students?

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Academy Goals: Mathematics in the Classroom Number Theory and Rational Numbers

Module A: Mathematical Literacy

The paraeducator will:

1. Identify common misconceptions about mathematics
2. Identify the role of communication in mathematical literacy development
3. Identify the goal of problem solving and its development in the classroom
4. Compare and contrast mathematical literacy and language/reading/writing (literacy) development

Module B: Number Theory and Number Representations

The paraeducator will:

1. Apply number theory concepts to represent numbers in a variety of ways
2. Develop working concepts of factors and multiples
3. Use place-value concepts to represent numbers
4. Develop meanings for common rational and irrational numbers

Module C: Number Sense and Fractional Concepts

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1. Use number sense to justify the reasonableness of solutions for a variety of computation and problem-solving situations
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Module D: Decimal and Percentage Concepts

The paraeducator will:

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Module E: Making Connections

The paraeducator will:

1. Develop conversion algorithms among fractions, decimals, and percentages
 2. Compare rational and irrational numbers through equivalent forms using a variety of strategies
 3. Apply computational procedures for fractions, decimals and percentages to real-world problems
-



Question 1: Common Responses

- Word problems (problem solving)
- Numbers
- Rules
- Problems
- Memorization
- Skills
- Drill
- Homework
- Not fun
- Too hard



Question 2: Common Responses

- Didn't get the "basics"
- Vocabulary
- Too many processes
- "Word problems"
- Not interested



Response Chart 3

Your Perceived Abilities

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Arithmetic:

Calculations Involving Predefined Rules



Mathematics is ...

1. a study of patterns and relationships
2. a way of thinking
3. an art
4. a language
5. a tool

- *Reys, Suydam, and Lindquist (1992)*



Learning a Second Language

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Math as a Language

A thick, horizontal yellow brushstroke that tapers at both ends, serving as a decorative underline for the title.

To learn the math language students/learners must:

1. Read it
2. Write it
3. Speak it
4. Do it



Math Journal



▲ Reflections

▲ Problem Solving

▲ Reference



Problem Solving: A Definition

Problem solving means engaging in a task for which the solution method is not known in advance.

In order to find a solution, students must draw on their knowledge, and through this process, they often develop new mathematical understandings.

Solving problems is not only a goal of learning mathematics, it is also a major means of doing so.

Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking.

NCTM (2000)



Problem Solving

Place the numbers 1-9 in a 3x3 grid so that each row, column and diagonal has the sum of 15. No number may be used more than once.



Skills and Concepts

- Addition rules
- Recognizing that the order doesn't matter when adding numbers; the answer is the same (commutative property of addition)
- Pattern development
- Organizing data



Problem Solving with a Plan

Steps for Problem Solving:

- Understand the problem
- Devise a plan
- Carry out the plan
- Look back



Comparing and Contrasting Reading and Math Literacy

Reading	Language	Writing



Defining Literacy

Reading Literacy	Mathematical Literacy
Fundamentals	Fundamentals
Rules	Rules
Sentence production to represent ideas	Mathematical sentences
Practical uses outside the classroom	Practical uses outside the classroom
Comprehension	Ability to explain a problem or process
Prediction	Patterns and problem solving
Communication	Communication



Module B

Instructor's Guide



Module B: Number Theory and Number Representations

A. Module Introduction

Use the transparency **Module Goals (T1)** to review the goals of the module.

Module B: Number Theory and Number Representations

The paraeducator will:

1. Apply number theory concepts to represent numbers in a variety of ways
2. Develop working concepts of factors and multiples
3. Use place-value concepts to represent numbers
4. Develop meanings for common rational and irrational numbers

Module B focuses on number theory, transparency and handout **Number Theory (T2/H1)**. *Number theory* is the study of non-zero whole numbers (counting numbers without zero) and their relationships. Mathematicians throughout history have been fascinated by the patterns and meanings of numbers. For example, the concept of lucky numbers and the popular fear of 13 come from number theory beliefs. Early elementary students begin to develop number theory patterns when they learn about evens and odds. Older students begin to break numbers down into their fundamental components. This allows students to understand how numbers and operations work. Number theory deepens the understanding of numbers and the results produced by computation.



Goal 1: Apply number theory concepts to represent numbers in a variety of ways.



1.1 Lecture: Multiplication and Factors



***Note to Instructor:** Remind participants to use their math journals to take notes during lecture periods.

Fifth-grade students are expected to have mastered whole-number concepts and begin to apply them to fractions and decimals. Later success with many math concepts depends on how well students understand multiplication properties. Understanding these properties goes beyond producing the traditional algorithms. This is a different level of understanding; this is number theory.

Beginning in grade five, students learn to break down numbers into factors. *Factors* are the numbers that multiply together to produce a *product* (see the transparencies **Number Theory [T2]** and **Factors and Products [T3]** and the handout **Number Theory [H1]**). From early elementary experiences, students know these concepts as multiplication facts. This simple idea forms the basis for many later, complex mathematical concepts.

As a review for the group, ask each person to write down all the ways to multiply two numbers (*factors*) to create the answer (*product*) of 12. Ask them to share the number of factors



for 12. Groups should come up with six factors: 1, 2, 3, 4, 6, and 12. Many groups will say three factors as they are thinking of the number of factor pairs, not individual factors.

Have groups also factor 9 or 25. Each of these only has three different factors as one factor is repeated. These special numbers will be discussed later.



1.2 Activity: Line ‘Em Up

The paraeducator will use factoring to look for patterns in the numbers 1-30.

Materials:

- Transparency **Line ‘Em Up Factor Chart (T4)**
- Transparency **Line ‘Em Up Observations (T5)**
- Calculators (optional)



1.2.1 Steps

- Introduce this activity as a continuation of the lecture. Explain that participants will look for patterns on the Line ‘Em Up chart to begin to look for number patterns and develop a base for number development.
- Place the transparency **Line ‘Em Up Factor Chart (T4)** on the overhead and explain that each number on the chart represents the number of *different* factors each number might have.
- Divide the class into groups of 4-6.
- Divide up the numbers 1-30 among the groups and ask each group to reach agreement on the number of *different* factors for each number. Remind them that a repeated factor is only counted once.
- Have the groups report their answers and write them on the transparency **Line ‘Em Up Factor Chart**. See if there are any challenges or corrections that need to be made. This chart should be replicated in the Problem Solving section of the paraeducators’ math journal. This chart will be used as a reference for several later activities. Key for instructor for transparency **Line ‘Em Up Factor Chart (T4)**.

		Number of Factors							
Number	1	2	3	4	5	6	7	8	9
	1	2	4	6	16	12		24	
		3	9	8		18		30	
		5	25	10		20			
		7		14		28			
		11		15					
		13		21					
		17		22					
		19		26					
		23		27					
		29							



- Ask participants to record any patterns they see on this chart in their journal. Share the patterns or observations as a class and record on the transparency **Line ‘Em Up Observations (T5)**. Emphasize that these observations only apply to the data found for the numbers 1-30.

Possible Observations:

- ▲ Only 1 is under category 1 – no other numbers will be included.
- ▲ All numbers except 2 are odd under category 2.
- ▲ All numbers under categories 6 and 8 are even.
- ▲ Category 2 includes the most factors for the numbers 1-30.
- ▲ There are no numbers under categories 7 or 9 (this will change past number 30).



1.3 Discussion: Define Prime and Composite Numbers

The paraeducator will classify numbers as prime or composite.



1.3.1 Steps

- In mathematics, we can classify each counting number as either *prime* or *composite*. Use the **Prime and Composite transparency (T6)**.
 - ▲ *Prime* – a number that has *exactly* two different factors – 1 and itself
- From the chart, ask where the *prime* numbers are located (they are in category 2).
- In their journal Reflections section, have participants define *prime* numbers and add any generalizations.
 - ▲ All primes are odd except 2
 - ▲ 2 is the only even prime
 - ▲ Factors are always 1 and the number
- Have participants define *composite* numbers in their own words. Share class observations to make sure all details are included.
 - ▲ Numbers that are not prime
 - ▲ Numbers with more than two different factors
 - ▲ May be odd or even numbers
- Generate some discussion about the number 1. If we use the definition of *composite* as “not prime,” 1 should be composite. The problem is that *composite* numbers have more than two *different* factors, which 1 does not. Students will then say that one is a *prime*. By definition, a *prime* number has *exactly* two *different* factors. The number 1 has two repeated factors. It doesn’t fit the prime definition either.
- For the number 1, we say it is *neither* prime *nor* composite. Remind the paraeducators to add this to their journal notes.
- Make sure the participants have included the words *exactly* and *different* in their *prime* definition and alter the *composite* definition to include “excludes 1.”



1.4 Lecture: Fundamental Theorem of Arithmetic



***Note to Instructor:** Remind participants to use their math journals to take notes during lecture periods.

The concepts of factors, primes and composite numbers are the foundation of the **Fundamental Theorem of Arithmetic (H2/T7)**.

Fundamental Theorem of Arithmetic:

Every counting number may be written as a distinct product of prime numbers (*prime factorization*)

The ability to break a number down into prime numbers allows for multiple analyses of composite numbers. For example, we can tell if one number divides another, if numbers have similar factors (used when reducing fractions), and we can break down complex problems into simpler, solvable parts.

Ask participants to break down the number 24 into the product of only prime numbers. Remind them that they already know factor pairs of 24. This should help them get started. Many participants will feel lost trying to figure out how to get started and organize the information. Several will use trial and error without any attention to the factors they already know (such as 6×4 , 8×3 , 12×2).

Answer: $24 = 2 \cdot 2 \cdot 2 \cdot 3$

According to the *Fundamental Theorem of Arithmetic*, $2 \cdot 2 \cdot 2 \cdot 3$ will only equal 24 as $2 \cdot 2 \cdot 2 \cdot 3$ is the unique *prime factorization*. Even if the order is changed, we still get 24. However, putting in any additional factor changes the product.

Larger numbers become cumbersome without some way to organize the factors. Two methods are commonly used for this purpose: tree diagrams and the division method (also called the “birthday cake” method). These two methods complete the same objective, but tend to be linked to certain learning styles. Usually students have a preference for which method works best for them.

Use 72 as an example for both methods.

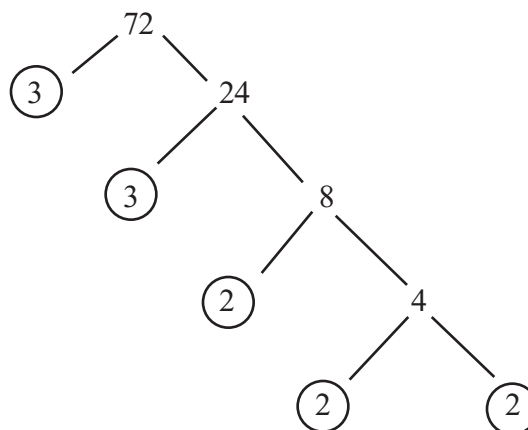
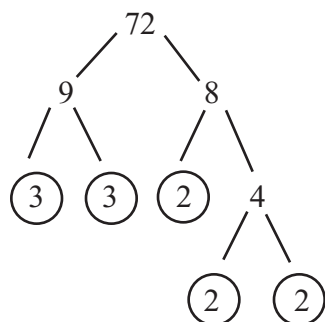
Tree Diagram

The tree diagram allows the learner to use any factor pair to begin, which cuts down on trial and error. The number is placed at the top, and the tree branches downward each time the tree “branches” into two limbs. A branch ends or “dies” when a prime number appears. Students can circle primes as they arise to help keep track of data. A student should know the basic prime numbers: 2, 3, 5, and 7. The prime factorization is the product of all the prime numbers used. There are several possible ways to do each tree but they all produce the same prime factorization.



Demonstrate the process below on a blank transparency.

Possible examples:



Each tree will produce the prime factorization of $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 72$.

Birthday Cake

This method makes the link between factoring and division. If a factor divides the number evenly, the quotient (answer) is the other factor. To find the prime factorization, the number is continuously divided by *only* prime numbers. Some students find this difficult because the numbers remain large longer than for the trees. The prime factorization is found on the outside. This method resembles a tiered cake because of the upward division.

Demonstrate the process below on a blank transparency. Any prime number may be used to start the process.

$$\begin{array}{r}
 36 \\
 2 \overline{) 72}
 \end{array}
 \qquad
 \begin{array}{r}
 18 \\
 2 \overline{) 36} \\
 2 \overline{) 72}
 \end{array}
 \qquad
 \begin{array}{r}
 9 \\
 2 \overline{) 18} \\
 2 \overline{) 36} \\
 2 \overline{) 72}
 \end{array}
 \qquad
 \begin{array}{r}
 3 \\
 3 \overline{) 9} \\
 2 \overline{) 18} \\
 2 \overline{) 36} \\
 2 \overline{) 72}
 \end{array}$$

Factorizations can get very long. *Exponents* can be used to shorten the writing process. *Exponents* (use transparency **Exponents [T8]**) are shorthand for multiplication. The factorization for 8 is $2 \cdot 2 \cdot 2$. This is commonly written as 2^3 , stated as “two to the third power” or “two cubed.” Two is called the *base*. The base is the number being multiplied. The *exponent* or *power* is the number of times the base is multiplied. A common error for exponents is to multiply the base and the exponent. An incorrect answer is $2^3 = 6$.

For the previous example, $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$ for 72 may be rewritten as $2^3 \cdot 3^2$. Note that each base is written separately, but the final answer is still a product of those bases ($2^3 \cdot 3^2 = 8 \cdot 9 = 72$). Exponents make it easy for students to analyze data and



use calculators. Exponents are a vital part of upper-level mathematics.

If time allows, have participants factor numbers in a method of their choice and write the prime factorization both as the expanded product and with exponents.



1.5 Discussion: Define the Order of Operations

The paraeducator will define and use the order of operations to simplify expressions.



1.5.1 Steps

- Use the transparency **Order of Operations Practice 1 (T9)**. Show only the problems; cover the answers. Have participants calculate the answers.
 1. $3 + 4 \times 5$
 2. $3 + 8 \div 2 \cdot 4$
- If calculators are available, have students enter the data. Most students will get an incorrect answer. Several assume it is a calculator error.
- Uncover the answers. Have students attempt to figure out how the calculator or the instructor got the answer and ask them to record their thoughts in their journal. Discuss the paraeducators' observations as a group and attempt to create some rules.
 1. 23
 2. 19
 - ▲ For #1, multiplication seems to come before addition
 - ▲ For #2, division comes first, then multiplication, and finally adding 3
- A vital math skill is the *order of operations*. The order of operations (see transparency **Order of Operations [T10]**) defines the order in which operations (addition, subtraction, etc.) are performed. Therefore, incorrect answers result from using an incorrect order of application.
- Many participants have learned Please Excuse My Dear Aunt Sally (PEMDAS) as a mnemonic to assist with the order of operations. Each letter represents a specific operation.
 - ▲ P – parentheses (or any enclosure symbol)
 - ▲ E – exponents
 - ▲ M – multiplication
 - ▲ D – division
 - ▲ A – addition
 - ▲ S – subtraction
- While this method is helpful, it actually teaches the order incorrectly if used literally.
- The mnemonic misleads the order by implying that multiplication comes before division; this is false.
 - ▲ P and E are correct – they must come first
 - ▲ Multiplication and division always come before addition and subtraction
 - ▲ For multiplication and division, they are performed from left to right as they occur (group them on the transparency **Order of Operations [T10]**).



- ▲ For addition and subtraction, they are performed from left to right as they occur (group them on the transparency **Order of operations [T10]**).
- Have participants complete the handout **Order of Operations 2 (H3)**. While solving, have participants write reasons why certain operations were performed in a specific order.
 1. $83 - 7 \cdot 6$ Answer: 41 Mult. comes before subtraction.
 2. $90 - 5 \cdot 5 \cdot 2$ Answer: 40 Mult. comes before sub.; left to right for mult.
 3. $2^2 + 5^2$ Answer: 29 Exp. come before add.
 4. $12 + (6 + 4)$ Answer: 22 Parentheses come before addition.
- When working the problems for participants on the transparency **Order of Operations Practice 2 (T11)**, the problem should be worked downward with one change per line.
 - $83 - 7 \cdot 6$
 1. $83 - 42$
 $= 41$
 - $90 - 5 \cdot 5 \cdot 2$
 $90 - 25 \cdot 2$
 2. $90 - 50$
 $= 40$
 - $2^2 + 5^2$
 $4 + 5^2$
 3. $4 + 25$
 $= 29$
 - $12 + (6 + 4)$
 4. $12 + 10$
 $= 22$



Goal 2: Develop working concepts of factors and multiples.



2.1 Lecture: Relationships Between Factors and Multiples



***Note to Instructor:** Remind participants to use their math journals to take notes during lecture periods.

One important relationship in number theory is that between *factor* and *multiple*. If one number is a factor, it must divide the product it creates. For example, if 3 is a factor of 12, then 3 must divide 12 evenly.

$$3 \overline{)12} = 4$$



We would say that 12 is a *multiple* of 3.

A *multiple* of a number is the product of that number and any whole number. There are an infinite number of multiples for every counting number.

Look at the following pattern on the transparency/handout **Multiples (T12/H4)**.

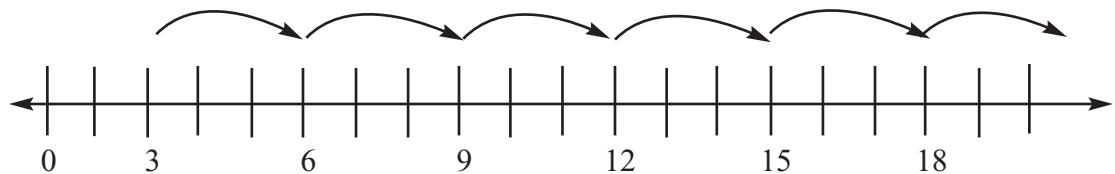
4, 8, 12, 16, 20, 24, 28 ...

Participants should recognize that the sequence increases by adding four each time. If we look at the first term, 4, another pattern is multiplication.

$4 \times 1, 4 \times 2, 4 \times 3, 4 \times 4, 4 \times 5, 4 \times 6, 4 \times 7, \dots$

For each product, the *factors* are listed in the above pattern.

Multiples may be identified on a number line. This is a good way to involve the relationship between addition and multiplication. The pattern for multiples of three would start at 3, and then increase by three “moves” on the number line.



The first few multiples of 3 are 3, 6, 9, 12, 15, and 18. The included factors are $3 \times 1, 3 \times 2, 3 \times 3, 3 \times 4, 3 \times 5, 3 \times 6, 3 \times 7, \dots$



2.2 Discussion: Developing Greatest Common Factor and Least Common Multiple

The paraeducator will define uses and methods for greatest common factor and least common multiple.



2.2.1 Steps

- Up to this point, factors and multiples have been applied to individual numbers. In mathematics, we use these skills to compare numbers and advance calculations.
- Two concepts for comparing numbers are *greatest common factor (GCF)* and *least common multiple (LCM)*. Many studies show that these are commonly confused because the processes used for comparison are similar. More important than calculation is that students understand the purpose of each concept. Several methods are available for calculation, but only a few will be shown here.
- Before working mathematically, using regular word meanings assists learners in understanding the concept. Discuss these words as a group. Fill in the



Comparing GCF and LCM transparency/handout (**T13/H5**) throughout the following discussions. The completed chart is listed here but should not be presented at the beginning. Fill in the chart as the lesson progresses.

- ▲ Greatest Common Factor (GCF): *greatest* means largest; *common* means same or shared; factor means the number that gets multiplied.
- ▲ Least Common Multiple (LCM): *least* means smallest; *common* means same or shared; *multiple* means product or multiply.

GCF	LCM
<ul style="list-style-type: none">• Uses factors• Implies division• Number is smaller than the given numbers• Divides the given numbers evenly (factors)• Listing algorithm: choose largest factor• Prime numbers: choose product of all common primes• Useful in problems finding equal groupings	<ul style="list-style-type: none">• Uses multiples• Implies multiplication• Number is larger than or equal to given numbers• May be divided evenly by given numbers (multiple)• May use listing algorithm or prime numbers• Listing algorithm: choose first common multiple• Prime numbers: choose product of the “greatest” number of different factors present in each number• Useful in problems finding common events

- Another important concept before beginning mathematically is to determine the type of number for which we are searching. Discuss this as a group and add to the chart.
 - ▲ *GCF* (also know as GCD – *greatest common divisor*) deals with factors. Factors are the numbers multiplied together to get a product. This implies that the GCF must be a number *smaller* than the original numbers. The term “divisor” says that the number must divide into the products evenly.
 - ▲ *LCM* uses multiplication, which means that numbers get larger. Therefore, this implies that the LCM must be a number *larger* or equal to the original numbers.

Greatest Common Factor

- Use the **Greatest Common Factor** handout/transparency (**H6/T14**). Beginning with GCF (GCD), we can use the very basic *listing algorithm*. For comparison, use 12 and 36. GCF needs the *largest factor* that will go evenly into 12 and 36. Allow participants to list all factors of each number and see if they can find the GCF. Hint: putting the factors in number order makes comparison easier.



- ▲ Factors of 12: 1, 2, 3, 4, 6, 12
- ▲ Factors of 36: 1, 2, 3, 4, 6, 9, 12, 18, 36
- ▲ Factors in common: 1, 2, 3, 4, 12
- ▲ The greatest factor they have in common is 12, so $\text{GCF}(12, 36) = 12$
Note: There is always a common factor for whole numbers. Often, the only common factor is 1.
- A practical example of GCF is in the following example. Have participants discuss their thoughts for student solutions to this problem.

Bo's aunt donated 12 cans of juice and 36 fruit snacks to his class for a party. Each student is to receive the same number of cans of juice and the same number of fruit snacks. What is the largest number of students that can be in Bo's class and share the food equally?

- ▲ Many students easily see this as division. That should be a clue that this is GCF.
- ▲ Some students incorrectly try to add the objects and then divide by a random number to produce equal groups.
- ▲ The problem is not asking how many total pieces of food each gets. It states that each student will get an equal number of juice cans and an equal number of fruit snacks.
- ▲ Students often begin to divide numbers into each total. This is a good place to start, and it captures the concept of division.
- ▲ Once students have attempted the problem and possibly gotten an answer via trial and error (12 students in the class), the teacher can show students how to organize their thinking using GCF from the work above.
- ▲ Some attention should be paid to other "common" factors such as 6, to show that they actually work but are not the greatest number of students.
- ▲ An extension of this activity is to ask how many cans of juice and fruit snacks each of the 12 students will get (each student will get 1 can of juice and 3 fruit snacks). This is found with basic division using factors.
- Another method for GCF used in middle-school grades involves prime numbers.
 - ▲ Have participants use a method of their choice to find the prime factorization for 12 and 36.

$$12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$$

$$36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$$
 - ▲ 12 was determined to be the GCF from the listing algorithm. Ask participants to analyze the prime factorization to see if they can create 12.
 - ▲ The concept of common factor is necessary here. Each product has a common 2, 2, and 3. This product equals 12.

Least Common Multiple

- Use the **Least Common Multiple** handout and transparency (**H7/T15**). The *listing algorithm* may be used to determine the LCM. This time, the list includes *multiples* of numbers. LCM needs the *smallest* (least) common



product (multiple) between the two numbers. Allow participants to list the multiples of 6 and 8 to see if they can find the LCM.

- ▲ Multiples of 6: 6, 12, 18, 24, 30, 36
Multiples of 8: 8, 16, 24, 32
- ▲ The first product in common is 24. So $\text{LCM}(6,8) = 24$
- A practical example of LCM is found in the following example. Have participants discuss their thoughts for how students would solve this problem.

Amy and Jose both work at night. Amy has every sixth night off and Jose has every eighth night off. If they are both off tonight, how many nights will it be before they are both off again?

- ▲ Students commonly want to divide or find the average of the numbers, which is incorrect.
- ▲ Any problem asking about a commonly occurring event (meeting, passing, working together, etc.) is a hint at trying to find the LCM.
- ▲ Students can work this problem by simply making a chart or number line of the event. This is an effective method because it targets the concept of multiplication.
- ▲ Once students have attempted the problem and possibly gotten an answer via trial and error (night 24), the teacher can show students how to organize their thinking using LCM as above.
- ▲ An extension of this activity could be to ask when the next time Amy and Jose will both be off. (night 48—the next multiple)
- Another method for LCM used in middle-school grades involves prime numbers. Remind paraeducators of the goal to find a multiple. In this case, both numbers should divide the common multiple; this makes the numbers into factors.
 - ▲ Have the participants use a method of their choice to find the prime factorization for 6 and 8.
$$6 = 2 \cdot 3$$
$$8 = 2 \cdot 2 \cdot 2 = 2^3$$
 - ▲ 24 was determined to be the LCM from the listing algorithm. Ask the class to use their knowledge to see if they can create 24.
 - ▲ Students usually find this more difficult to see. A good visual here is to treat each factor as a playing card. The 6 has a 2 card and a 3 card. The 8 has three 2 cards. Each factor must be in the final multiple for it to divide evenly.
 - ▲ The task is to find how many of each factor to keep. Sticking to the card game scenario, ask who has the winning hand of each factor.
 - 8 has the winning hand of 2's – so we take its three 2's
 - 6 has the winning hand of 3's – so we take its 3
 - This gives an $\text{LCM}(6,8) = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3 = 24$
 - ▲ This method takes much practice, but is more efficient than the listing method for large numbers or multiple number comparisons of three or more products.
- Finish filling in the chart as a class. See the completed chart listed earlier in this goal.



Goal 3: Use place-value concepts to represent numbers.



3.1 Lecture: Review Place Value for Whole Numbers



***Note to Instructor:** Remind participants to use their math journals to take notes during lecture periods.

As students in grades 5-8 move away from whole-number concepts into decimals and fractions, many concepts will depend on their understanding of basic place value for whole numbers.

Remind participants that our number system is base-10, using the **Base-10** handout and transparency (**H8/T16**).

- Each place value in a number represents a certain power of 10.
- Each digit tells how many groups of a certain power of 10.
- Digits go from 0-9.
- Regrouping occurs when 10 is reached in any place value.

For whole numbers, students should come to fifth grade able to identify basic place value. For 1,234,

1	2	3	4
Thousands	Hundreds	Tens	Ones

Review that place value starts at the right with the ones place and increases by powers of 10. If available, show a simple example such as 231 with overhead base-10 blocks (2 flats, 3 rods, and 1 unit). These are common manipulatives in the elementary classroom. Students who can visualize a number with base-10 blocks find it easier to understand the decimal concepts.



***Note to Instructor:** If you do not have base-10 blocks available, use the **Paper Base 10 Blocks** handout (**H9**) and cut out enough of the paper blocks necessary for your class. Run a copy on overhead plastic and cut out for use on the overhead.

Ask the participants to predict the next place value to the left to see if they understand the pattern. (ten thousands)

Students should be able to write any whole number in *expanded notation* from grades K-4. *Expanded notation* is the sum of the values of the place value digits. Use the **Expanded Notation** transparency (**T17**).



- $1,234 = 1 \times 1000 + 2 \times 100 + 3 \times 10 + 4 \times 1 = 1000 + 200 + 30 + 4$
- This says one group of 1,000, two groups of 100, three groups of 10 and four groups of 1.

This is a good opportunity to link new knowledge about exponents to place value. Starting with basic whole numbers creates a foundation for decimals. Each place value may be written as a product of 10, hence the *base-10* system.

$$1,234 = 1 \times (10 \times 10 \times 10) + 2 \times (10 \times 10) + 3 \times 10 + 4 \times 1$$

Use exponents to shorten the process. This shows the increasing size of the powers of 10.

$$1,234 = 1 \times 10^3 + 2 \times 10^2 + 3 \times 10^1 + 4 \times 1$$

Participants should note the increasing exponents as the place value gets larger.



***Note to Instructor:** Some discussion may arise about why the ones place does not have a power of 10. According to the pattern, it would appear that there should be 10^0 with the 4. This is actually mathematically correct because any number to the zero power (except zero to the zero power) equals one. So, $4 \times 10^0 = 4 \times 1$. This is a skill beyond the scope of this course, but the instructor can share this as something to research further.

An easier answer to the above note is that the ones place does not have a 10. Once a 10 exists, it rolls over into the tens place. The ones represent those that have not yet made it to a 10. They represent single units or ones.



3.2 Discussion: Applying Expanded Notation to Decimals

The paraeducator will apply whole-number place value knowledge to write decimals in expanded notation.



3.2.1 Steps

- After the whole-number place value review, participants should transition easily into decimal notation.
- Review decimal place value using the **Decimal Place Value** transparency and handout (**T18/H10**).
 - ▲ Decimals are also based on the base-10 system.
 - ▲ Numbers to the left of the decimal represent whole numbers.
 - ▲ Numbers to the right of the decimal represent values less than 1 (also thought of as fractional values).
- The decimal acts as the notation to signal whole numbers and fractional numbers. For example, 2.761

2	.	7	6	1
Ones		Tenths	Hundredths	Thousandths



- ▲ Ask participants to predict the next place values to the right. (ten thousandths)
- As a group, discuss similarities and differences between decimal place value and whole-number place value. Record the discussion ideas on the **Place-Value Comparison** transparency (T19) and also make sure participants record them in their personal journals.
- ▲ See the chart below for possible responses.

Whole-Number Place Value	Decimal Place Value
<ul style="list-style-type: none"> • Based on powers of 10 • Represents whole values • Uses same base words of “ten,” “hundred” and “thousand” • Includes place for ones • Value gets larger moving left 	<ul style="list-style-type: none"> • Based on powers of 10 • Represents fractional values • Uses same base words of “ten,” “hundred” and “thousand” • All words end in “ths” • No “oneths” place • Value gets smaller moving right

- Participants struggle with the words because “thousand” is typically larger than “hundred” except with decimals.
- If using place value blocks, remind participants that the value of each block changes for decimals. The one whole in this case is a $10 \times 10 \times 10$ cube (or 10 flats stacked on top of one another or 1000 small units). That makes the single “unit” a thousandth. See if the group can define the flat and rod.
 - ▲ Flat – 1 tenth
 - ▲ Rod – 1 hundredth
 - ▲ Unit – 1 thousandth
- On the overhead, write the number 0.235 (2 flats, 3 rods, 5 units if using blocks).
 - ▲ Have participants say the number out loud. (two hundred thirty-five thousandths)
 - ▲ Using what they know about place value, have participants say aloud each place value. (two tenths, three hundredths, 5 thousandths)
- We know that with expanded notation, the sum of the products must give the final answer of 0.235. Discuss possible ways to complete this task.
 - ▲ Some participants will be able to see the decimals connection immediately $0.235 = 2 \times 0.1 + 3 \times 0.01 + 5 \times 0.001$.
 - ▲ Others will need to see the fractional equivalent as their knowledge is based on powers of 10. Go back to the prior step of saying each decimal place value aloud.

$$2 \text{ tenths} = 2 \times \frac{1}{10}$$

$$3 \text{ hundredths} = 3 \times \frac{1}{100}$$

$$5 \text{ thousandths} = 5 \times \frac{1}{1000}$$

$$\text{So } 0.235 = 2 \times \frac{1}{10} + 3 \times \frac{1}{100} + 5 \times \frac{1}{1000}$$
 (expanded notation in fractions)



- ▲ Participants should begin to again see the connection to base-10 place-value concepts of whole numbers. Participants should understand that the larger the number divided by, the smaller the number becomes.

- ▲ Link this back to exponents to show the pattern of exponents

$$0.235 = 2 \times \frac{1}{10} + 3 \times \frac{1}{(10 \times 10)} + 5 \times \frac{1}{(10 \times 10 \times 10)} = 2 \times \frac{1}{10^1} + 3 \times \frac{1}{10^2} + 5 \times \frac{1}{10^3} = \frac{2}{10^1} + \frac{3}{10^2} + \frac{5}{10^3}$$

- ▲ Participants should be able to see the initial decimal equivalent because the denominator tells how many places over the number it actually sits.



***Note to Instructor:** In seventh and eighth grade, students learn to use negative exponents instead of fractions. For this discussion, we will only be using fractional notation.



Goal 4: Develop meanings for common rational and irrational numbers.



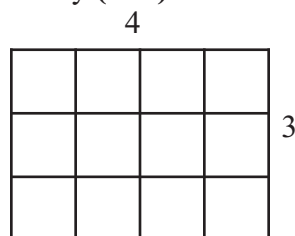
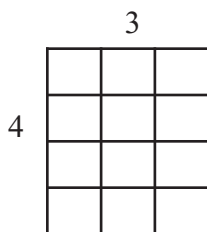
4.1 Discussion: Perfect Squares

The paraeducator will define whole numbers that are perfect squares.



4.1.1 Steps

- Provide each participant with the **Centimeter Grid Paper** handout (H11).
- Remind participants of multiplication facts and the area model. The fact pair would look like one of the following and have an area of 12 (12 squares); use the **Centimeter Grid Paper** transparency (T20).



- Have participants draw three shapes that make squares rather than rectangular shapes with any number of squares (area). Record their area and dimensions.
 - ▲ Most will get 4 (2 x 2), 9 (3 x 3), 16 (4 x 4), or 25 (5 x 5).
- Redirect participants to the **Line 'Em Up Chart (T4)** created in Goal 1. Ask what they notice.
 - ▲ These special numbers fall under the odd number of factors categories.
 - ▲ They have a repeated factor.
- Ask what other categories might contain perfect squares. (7 or 9 because of repeated factors making an odd number of factors)
- Numbers such as 4, 9, 16, and 25 are called *perfect squares*. Perfect squares have a factor pair that has identical factors (show transparency **Perfect Squares [T21]**). This implies they can create a square shape with equal sides.



- Returning to the exponent concept, see if participants can use the dimensions listed above to show each perfect square as a number to the second power (also called *squared*)
 - ▲ $2 \cdot 2 = 2^2 = 4$
 - ▲ $3 \cdot 3 = 3^2 = 9$
 - ▲ $4 \cdot 4 = 4^2 = 16$
 - ▲ $5 \cdot 5 = 5^2 = 25$
- Note that each factor is raised to the second power. We say these as “two squared,” “three squared,” and so on, because they create a square when drawn.
- Ask participants to predict the next perfect square. (36)
 - ▲ Find the factor pairs of 36 to see if it would be under an odd number of factors. (has 9 factors)
 - ▲ Discuss the pattern of increasing consecutive numbers listed above: 2, 3, 4, 5, 6. (36 came from 6×6 , as 6 was next in the pattern)
 - ▲ Discuss whether there is a smaller perfect square than 4. According to the pattern, $1 \cdot 1$ should be the first pattern.
 - ▲ Can a square be drawn that is 1×1 or one unit by one unit? (Yes – it is a single square, so 1 is a perfect square)
 - ▲ 1 fits the conjecture of odd factors discovered earlier.



4.2 Lecture: Square Roots and Irrational Numbers

Early elementary-grade students already have developed concepts of inverse processes: addition with subtraction and multiplication with division. Students in grades 5-8 add several more concepts to their knowledge of inverse processes. The new inverse processes introduced here include squaring and taking the *square root*. Squaring a number implies taking the same number (equal sides of a square) and multiplying them to produce a perfect square (product). There must be a way to start at the product and return to the side length (or repeated factor); this is the *square root*; use transparency and handout **Square Roots (T22/H12)**.

With perfect squares, we know that the side length will be a whole number. The *radical symbol* ($\sqrt{\quad}$) implies, “Given a certain area, find the side length of the square.” For example, $\sqrt{4}$ implies, “What is the side length of a square for an area of 4?” We know from our experience that the square is created from sides of 2. So $\sqrt{4} = 2$. Two is called the *root*. We can check by squaring the two; we get an answer of 4.

In later grades, it is possible to have fractional answers although it is difficult to explain with the drawing concepts learned here. It is important to remember that the *radical symbol* ($\sqrt{\quad}$) is only asking for the repeated factor. Whole number answers or fractional answers are referred to as *rational numbers*. This will be discussed in the next module.

Returning to whole-number concepts, we know from the **Line ‘Em Up Chart** that not all numbers are perfect squares. For example, $\sqrt{2}$ asks, “What is the side length of a square for an area of 2?” We know from our drawings that we cannot draw a square using two units,



so the side length cannot be a whole number. In other words, there are not two identical whole-number factors that can multiply to give a product of 2. If put into the calculator, $\sqrt{2} \approx 1.4141\dots$. This *non-terminating* (never ending), *non-repeating* decimal tells us there is no exact identical factor for the side of the square with an area of two; this number is *irrational*. Use transparency/handout **Irrational Numbers (T23/H13)**.

In mathematics, much attention is paid to classifying numbers. Early elementary students work on the first two categories: *natural* numbers (counting numbers starting with 1) and the *whole* numbers (natural numbers and zero). In grades 5-8 the categories begin to broaden: *rational numbers* and *irrational numbers*. A number cannot be both rational and irrational. This concept is dealt with in secondary grades.



4.3 Discussion: Estimating Irrational Numbers

The paraeducator will estimate the value of irrational numbers based on the prior knowledge of perfect squares.



4.3.1 Steps

- Knowing vocabulary and definitions is important, but students must be able to integrate this knowledge into their developing number sense.
- Students often do not have calculators in grades 5-8 and must rely on their concrete experiences to develop a mathematical equivalent.
- *Rational numbers* (those that are whole or fractional) provide an exact answer that may easily be placed on a number line or drawn. Use the transparency **Rational and Irrational Numbers (T24)**.
- *Irrational numbers* are always approximations because the decimal never ends.
- Ask participants to refer back to their centimeter graph paper where they first drew squares.
 - ▲ Ask what happened each time the side length got larger. (the area or product also got larger)
 - ▲ Ask what is expected for continuously growing side lengths. (the area or product will also continue to grow)
 - ▲ These observations are what is required to estimate roots.
- Ask participants to approximate the value of the irrational number, $\sqrt{5}$. Use the handout/transparency **Estimating Roots (H14/T25)**.
 - ▲ Have participants record their thoughts in their journal along with their approximations.
 - ▲ To approximate a root, find what two other perfect squares the number falls between.
 - ▲ It was shown that $\sqrt{4} = 2$.
 - ▲ The next perfect square is 9; so $\sqrt{9} = 3$.
 - ▲ The $\sqrt{5}$ is between $\sqrt{4}$ and $\sqrt{9}$, but must be closer to $\sqrt{4}$.
 - ▲ $\sqrt{6}$ is a little less than half way between 2 and 3, about 2.4.
 - ▲ $\sqrt{5}$ should be approximately 2.2 (it is actually 2.23606...).



- While the notation may look complex, the skill depends completely on whole-number concepts mastered in early elementary grades



4.4 Activity: Are You Irrational?

The paraeducator will identify rational and irrational numbers and approximate when necessary.

Materials:

- Handout **Are You Irrational? (H15)**
- Handout **Centimeter Grid Paper (H11)**



4.4.1 Steps

- In pairs, have participants determine whether they can draw a square with the given area.
 - ▲ If so, evaluate the square root to verify the side length.
 - ▲ If not, approximate the decimal value.
 - ▲ Each problem should be labeled *rational* or *irrational*.
- Go over results as a class to check on group progress.
- Answer:
 1. square is 2×2 ; side length of 2; rational
 2. not possible; $\sqrt{7} = 2.6$; irrational
 3. not possible; $\sqrt{15} = 3.9$; irrational
 4. square is 6×6 ; side length of 6; rational
 5. not possible; $\sqrt{24} \approx 4.9$; irrational

5.1. Assignment #1: Sieve of Eratosthenes



***Note to Instructor:** You will need to decide how much time to give the class to complete assignments so that you have time to grade, record grades and turn in materials from this course in a timely manner. If paraeducators are taking this course for credit, there will be a time limit based upon the grading period at the attending institution. You will also have to decide how you would like attendees to turn in their assignments (e.g., mailing them to you, dropping them off at your office). **You are strongly encouraged to be very firm about a completion date and may need to make some effort to follow up on attendees and their progress. Refer to the Grading Rubric handout (GR) for details on grading.**

Distribute handout **Assignment #1: Sieve of Eratosthenes (H16)**. Read the instructions and answer questions regarding completion of the assignment. Provide the class with a date for completion and your process for handing the assignment in. To assist in grading, answers to the assignment questions are provided below.



Assignment #1: Sieve of Eratosthenes

The assignment is worth 100 points.

There are two parts.

The focus of this assignment is to further explore number theory concepts of prime and composite numbers, and multiples and factors.

Part 1: (Total 50 points)

Using the attached hundreds board:

1. Start with 2. Color in all of the multiples of 2 (leaving 2 uncolored).
2. Move to 3. Color all of the multiples of 3 (leaving 3 uncolored).
3. Move to the next uncolored number. Color its multiples, but leave the initial number blank.
4. Complete this for the entire chart.
5. If a number is already colored, skip that number and move to the next multiple in the sequence.

Part 2: (50 points)

Use your hundreds board results to answer the following questions:

1. List the numbers that are left uncolored on the board.
Answers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97
2. How would you classify these numbers using a term learned in Module B?
Answer: Primes
3. What is the purpose of the Sieve of Eratosthenes?
Answer: To identify prime numbers to 100.
4. Why is 1 crossed out on the chart?
Answer: 1 is not prime or composite.
5. Name 2 patterns that you noticed while completing the chart.
Answers will vary.
 - a. Some numbers came under more than one list of multiples.
 - b. All of the numbers under 2, 4, 6, 8 are all colored in.
 - c. Many of the primes have something to do with 7.



Module B

Handouts



Number Theory

The study of non-zero whole numbers
(counting numbers without zero)
and their relationships.

Product

Answer to a multiplication problem.

Factors

Numbers that multiply to create a product

$$\text{factor} \times \text{factor} = \text{product}$$



Fundamental Theorem of Arithmetic

Every counting number may be written
as a distinct product of prime numbers
(*prime factorization*)

Prime Factorization

Product of prime numbers

$$24 = 2 \cdot 2 \cdot 2 \cdot 3$$

(prime factorization)



Order of Operations Practice 2

Complete each of the following using the order of operations. For each problem, explain your reasoning for the order decisions you made.

1. $83 - 7 \cdot 6$

2. $90 - 5 \cdot 5 \cdot 2$

3. $2^2 + 5^2$

4. $12 + (6 + 4)$



Multiples

A *multiple* of a number is the product of that number and any whole number.

4, 8, 12, 16, 20, 24, 28 ...

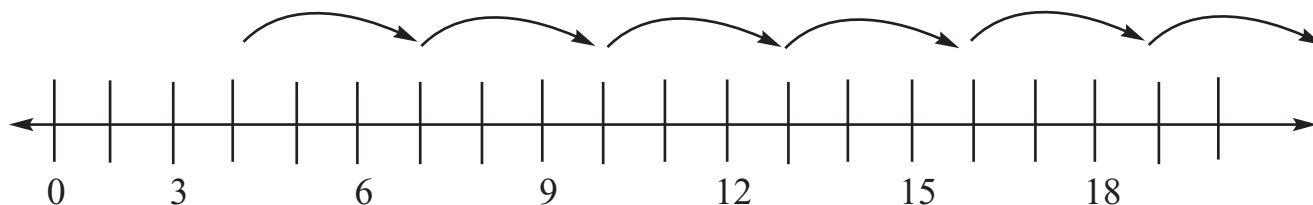
4×1 , 4×2 , 4×3 , 4×4 , 4×5 , 4×6 , 4×7 , ...

For each product, the *factors* are listed in the above pattern.

The first few multiples of 3 are 3, 6, 9, 12, 15, and 18. The included factors are

3×1 , 3×2 , 3×3 , 3×4 , 3×5 , 3×6 , 3×7 , ...

The pattern for multiples of three would start at 3,
and then increase by three “moves” on the number line.





Comparing GCF and LCM

Greatest Common Factor	Least Common Multiple



Greatest Common Factor

Find the factors of the following using the *listing algorithm*:

12 →

36 →

Find the GCF:

Try this:

Bo's aunt donated 12 cans of juice and 36 fruit snacks to his class for a party. Each student is to receive the same number of cans of juice and the same number of fruit snacks. What is the largest number of students that can be in Bo's class and share the food equally?

Find the factors of the following using *prime numbers*:

12 →

36 →

Find the GCF:



Least Common Multiple

Find the multiples of the following using the *listing algorithm*:

6 →

8 →

Find the LCM:

Try this:

Amy and Jose both work at night. Amy has every sixth night off and Jose has every eighth night off. If they are both off tonight, how many nights will it be before they are both off again?

Find the multiples of the following using *prime numbers*:

6 →

8 →

Find the LCM:



Base-10

- Each place value in a number represents a certain power of 10.
- Each digit tells how many groups of a certain power of 10.
- Digits go from 0-9.
- Regrouping occurs when 10 is reached in any place value.

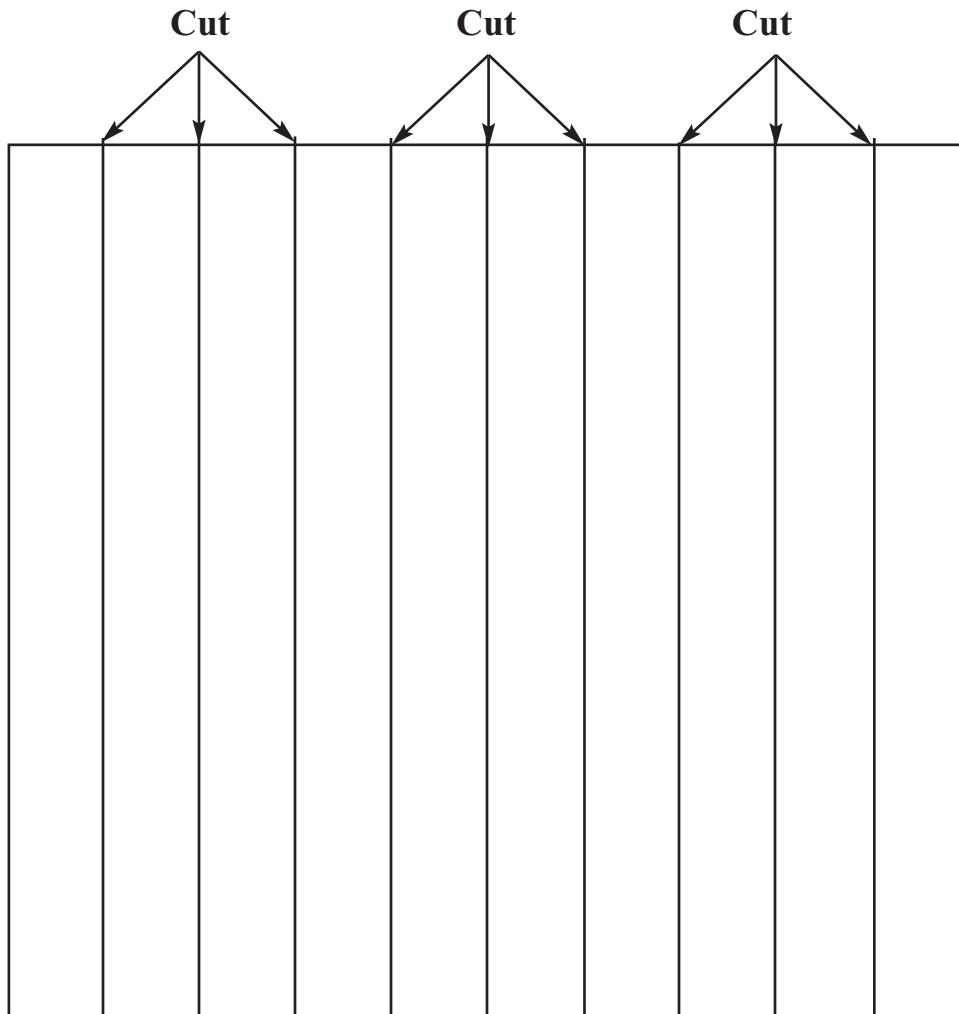
For 1,234:

1	2	3	4
Thousands	Hundreds	Tens	Ones



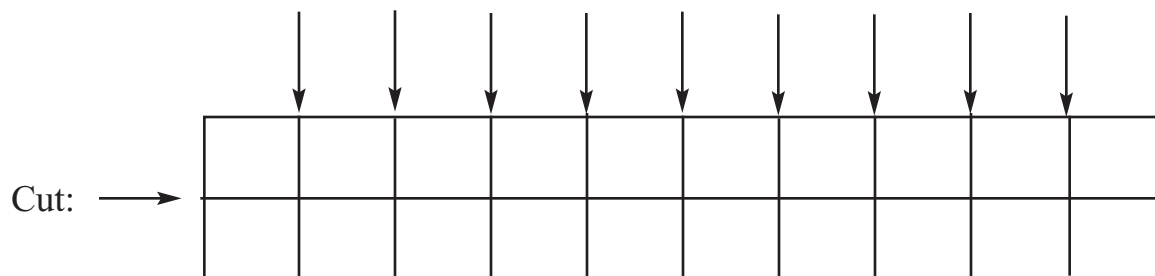
Base-10 Blocks

Longs:



Units:

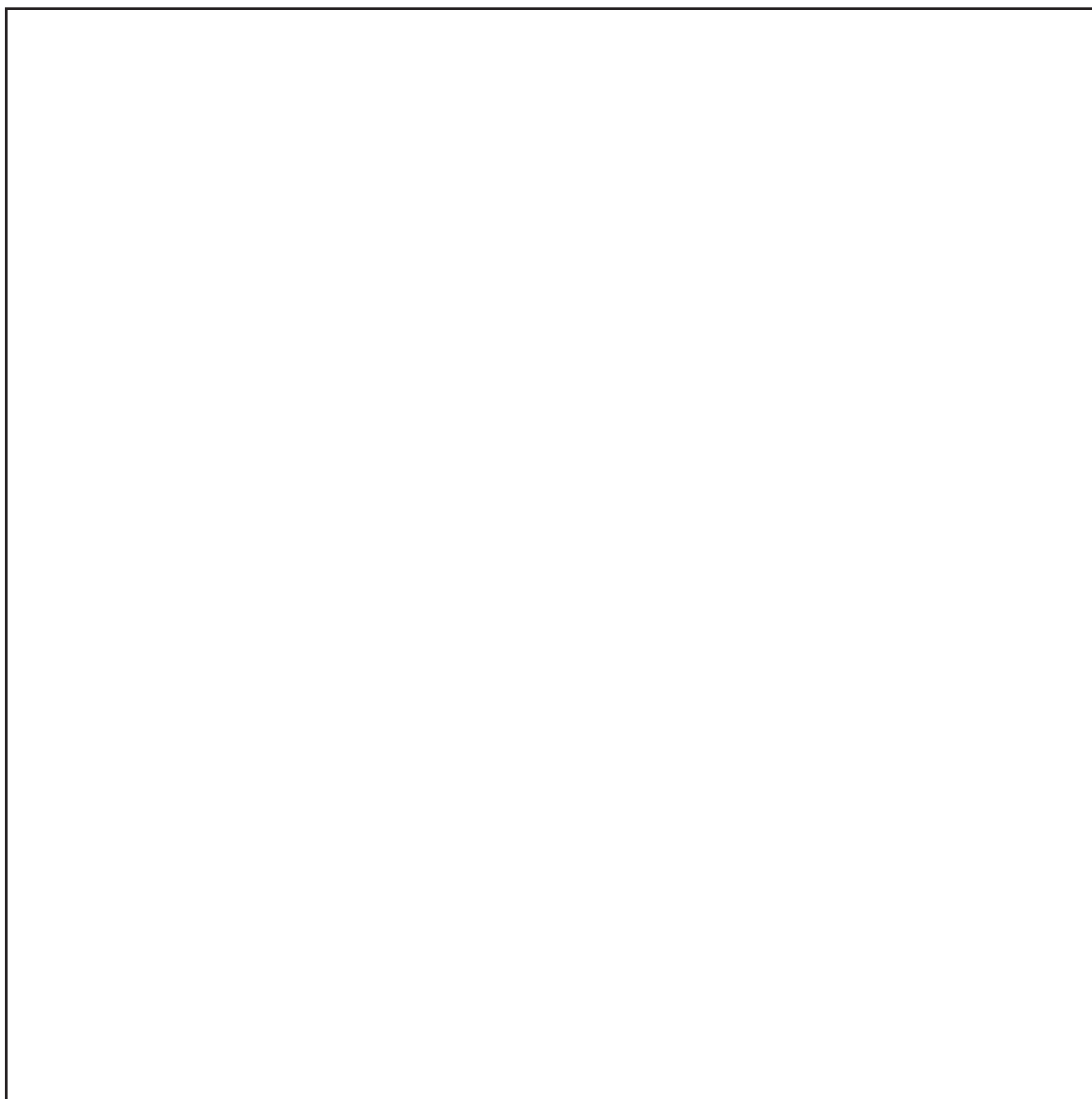
Cut on all cross lines:





Base-10 Blocks

Flat:





Decimal Place Value

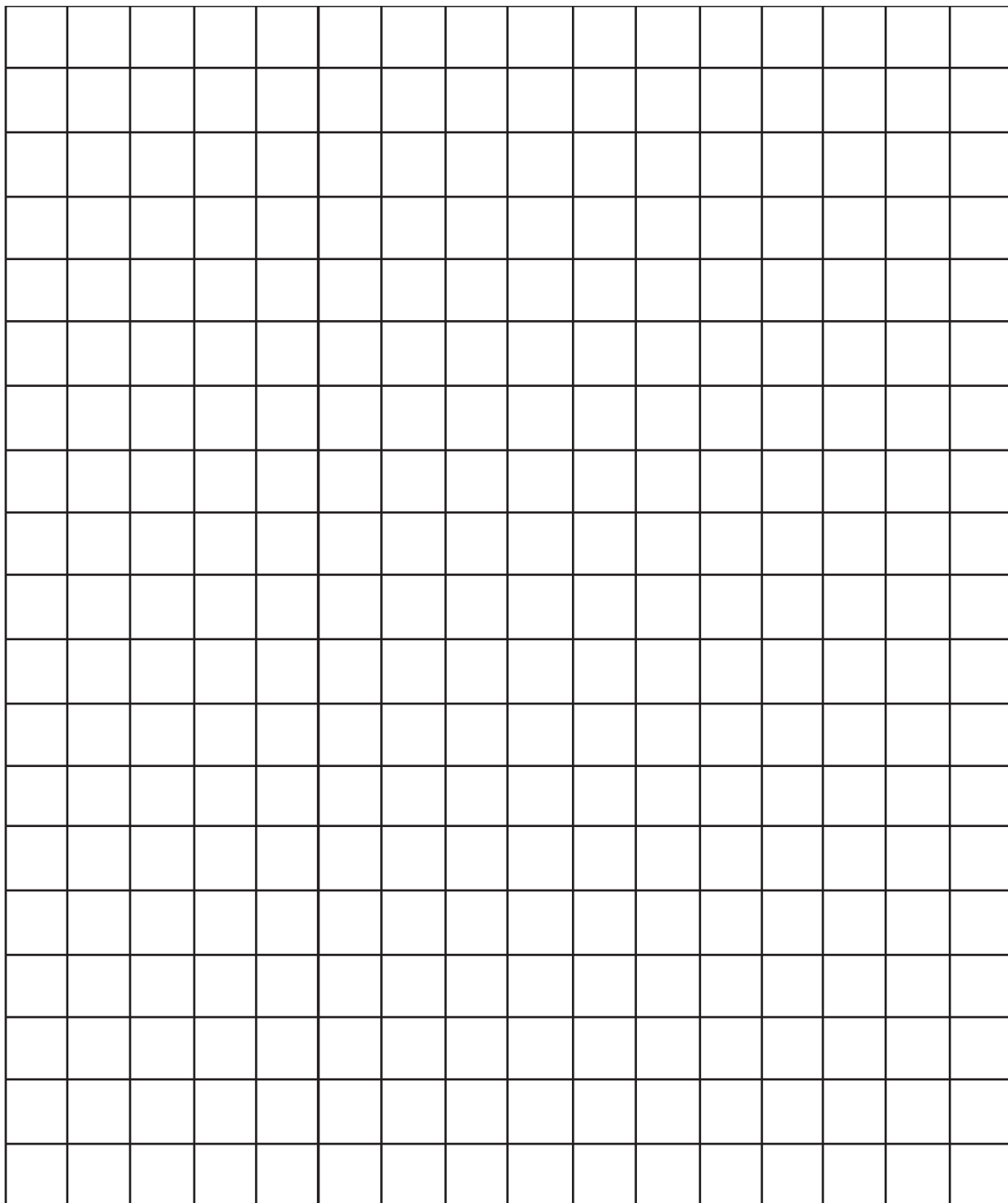
- Decimals are also based on the base-10 system.
- Numbers to the left of the decimal represent whole numbers.
- Numbers to the right of the decimal represent values less than 1 (also thought of as fractional values).

For 2.761:

2	.	7	6	1
Ones		Tenths	Hundredths	Thousandths



Centimeter Grid Paper





Square Roots

$$\begin{aligned}\text{Perfect Squares} \rightarrow 4 &= 2^2 \\ 9 &= 3^2 \\ 16 &= 4^2\end{aligned}$$

Radical Symbol $\sqrt{\quad}$

Asking for the repeated factor that created the product (area).

For a square, radical symbols ask for the side length of the square.

Square Root

$$\sqrt{4} = 2$$

$$\sqrt{9} = 3$$

$$\sqrt{16} = 4$$

Rational Numbers

Whole-number answers or fractional answers are referred to as *rational numbers*.



Irrational Numbers

***Irrational numbers are non-terminating
(never ending), non-repeating decimals.***

**For roots, it tells us there is no exact
identical factor for the side of the square to
create the given area.**



Estimating Roots

Try estimating $\sqrt{5}$

- To approximate a root, find what two other perfect squares the number falls between.
- It was shown that $\sqrt{4} = 2$.
- The next perfect square is 9; so $\sqrt{9} = 3$.
- The $\sqrt{5}$ is between $\sqrt{4}$ and $\sqrt{9}$, but must be closer to $\sqrt{4}$.
- $\sqrt{6}$ is a little less than half way between 2 and 3, about ≈ 2.4 .
- $\sqrt{5}$ should be approximately 2.2 (it is actually 2.23606 ...).



Are You Irrational?

Determine if you can draw a square for each area given. If so, draw the square and evaluate the root to find the side length. If not, approximate the root to the nearest tenth (one decimal). Label each problem as rational or irrational.

1. $\sqrt{4}$

2. $\sqrt{7}$

3. $\sqrt{15}$

4. $\sqrt{36}$

5. $\sqrt{24}$



Assignment #1

Name _____

Assignment# 1: Sieve of Eratosthenes

The following assignment is worth 100 points.

The focus of this assignment is to further explore number-theory concepts of prime and composite numbers, and multiples and factors.

There are two parts to this assignment.

Part 1: (50 points)

Using the attached hundreds board:

1. Start with 2. Color in all of the multiples of 2 (leaving 2 uncolored).
2. Move to 3. Color all of the multiples of 3 (leaving 3 uncolored).
3. Move to the next uncolored number. Color its multiples but leave the initial number blank.
4. Complete this for the entire chart.
5. If a number is already colored, skip it and move to the next multiple in the sequence.

Part 2: (50 points)

Use your hundreds board results to answer the following questions:

1. List the numbers that are left uncolored on the board.

2. How would you classify these numbers using a term learned in Module B?



100 Board

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100



Module B

Transparencies



Module Goals

Module B: Number Theory and Number Representations

The paraeducator will:

- **Apply number theory concepts to represent numbers in a variety of ways**
- **Develop working concepts of factors and multiples**
- **Use place-value concepts to represent numbers**
- **Develop meanings for common rational and irrational numbers**



Number Theory

**The study of non-zero whole numbers
(counting numbers without zero)
and their relationships.**



Factors and Products

Product

Answer to a multiplication problem.

Factors

Numbers that multiply to create a product.

$$\text{factor} \times \text{factor} = \text{product}$$



Line 'Em Up Factor Chart

9	
8	
7	
6	
5	
4	
3	
2	
1	



Line ‘Em Up Observations

1.

2.

3.

4.

5.

6.



Prime and Composite Numbers

Prime Numbers

A number that has *exactly 2 different* factors – 1 and itself.

Examples: 2, 3, 5, 7, 11, ...

Composite Numbers

Numbers that are not prime.

Numbers with more than two different factors.

Examples: 4, 6, 8, 9, 10, ...



Fundamental Theorem of Arithmetic

**Every counting number may be written as a
distinct product of prime numbers
(*prime factorization*)**

Prime Factorization

Product of prime numbers

$$24 = 2 \cdot 2 \cdot 2 \cdot 3$$

(prime factorization)

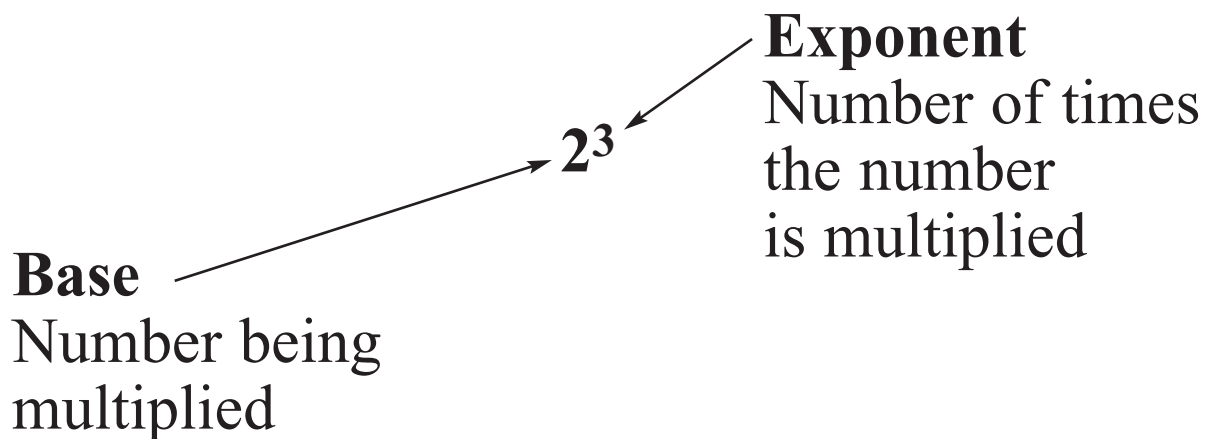


Exponents

Shorthand for multiplication

$$2 \cdot 2 \cdot 2 = 8$$

written with exponents as



Example:

$$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^3 \cdot 3^2 = 72$$



Order of Operations Practice 1

1. $3 + 4 \times 5$

2. $3 + 8 \div 2 \cdot 4$

Answers:

1. 23

2. 19



Order of Operations

Rules for simplifying expressions

PEMDAS (Please Excuse My Dear Aunt Sally)

- P – Parentheses (or any enclosure symbol)
- E – Exponents
- M – Multiplication
- D – Division
- A – Addition
- S – Subtraction



Order of Operations Practice 2

1. $83 - 7 \cdot 6$

2. $90 - 5 \cdot 5 \cdot 2$

3. $2^2 + 5^2$

4. $12 + (6 + 4)$



Multiples

A *multiple* of a number is the product of that number and any whole number.

4, 8, 12, 16, 20, 24, 28 ...

4×1 , 4×2 , 4×3 , 4×4 , 4×5 , 4×6 , 4×7 , ...

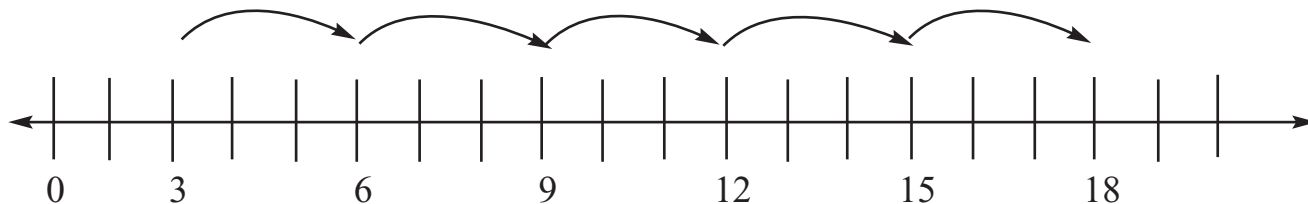
For each product, the *factors* are listed in the above pattern.

The first few multiples of 3 are 3, 6, 9, 12, 15, and 18 ...

The included factors are

3×1 , 3×2 , 3×3 , 3×4 , 3×5 , 3×6 , ...

The pattern for multiples of three would start at 3, and then increase by three “moves” on the number line.





Comparing GCF and LCM

Greatest Common Factor	Least Common Multiple



Greatest Common Factor

Find the factors of the following using the *listing algorithm*:

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Find the GCF:

Try this:

Bo's aunt donated 12 cans of juice and 36 fruit snacks to his class for a party. Each student is to receive the same number of cans of juice and the same number of fruit snacks. What is the largest number of students that can be in Bo's class and share the food equally?

Find the factors of the following using *prime numbers*:

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Find the GCF:



Least Common Multiple

Find the multiples of the following using the *listing algorithm*:

6 →

8 →

Find the LCM:

Try this:

Amy and Jose both work at night. Amy has every sixth night off and Jose has every eighth night off. If they are both off tonight, how many nights will it be before they are both off again?

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Base-10

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- Digits go from 0-9.
- Regrouping occurs when 10 is reached in any place value.

For 1,234:

1	2	3	4
Thousands	Hundreds	Tens	Ones



Expanded Notation

***Expanded notation* is the sum of the values of the place-value digits.**

$$\begin{aligned} 1,234 &= 1 \times 1,000 + 2 \times 100 + 3 \times 10 + 4 \times 1 \\ &= 1000 + 200 + 30 + 4 \end{aligned}$$

$$\begin{aligned} 1,234 &= 1 \times (10 \times 10 \times 10) + 2 \times (10 \times 10) + \\ &3 \times 10 + 4 \times 1 \end{aligned}$$

$$1,234 = 1 \times 10^3 + 2 \times 10^2 + 3 \times 10^1 + 4 \times 1$$



Decimal Place Value

- Decimals are also based on the base-10 system.
- Numbers to the left of the decimal represent whole numbers.
- Numbers to the right of the decimal represent values less than 1 (also thought of as fractional values).

For 2.761:

2	.	7	6	1
Ones		Tenths	Hundredths	Thousandths

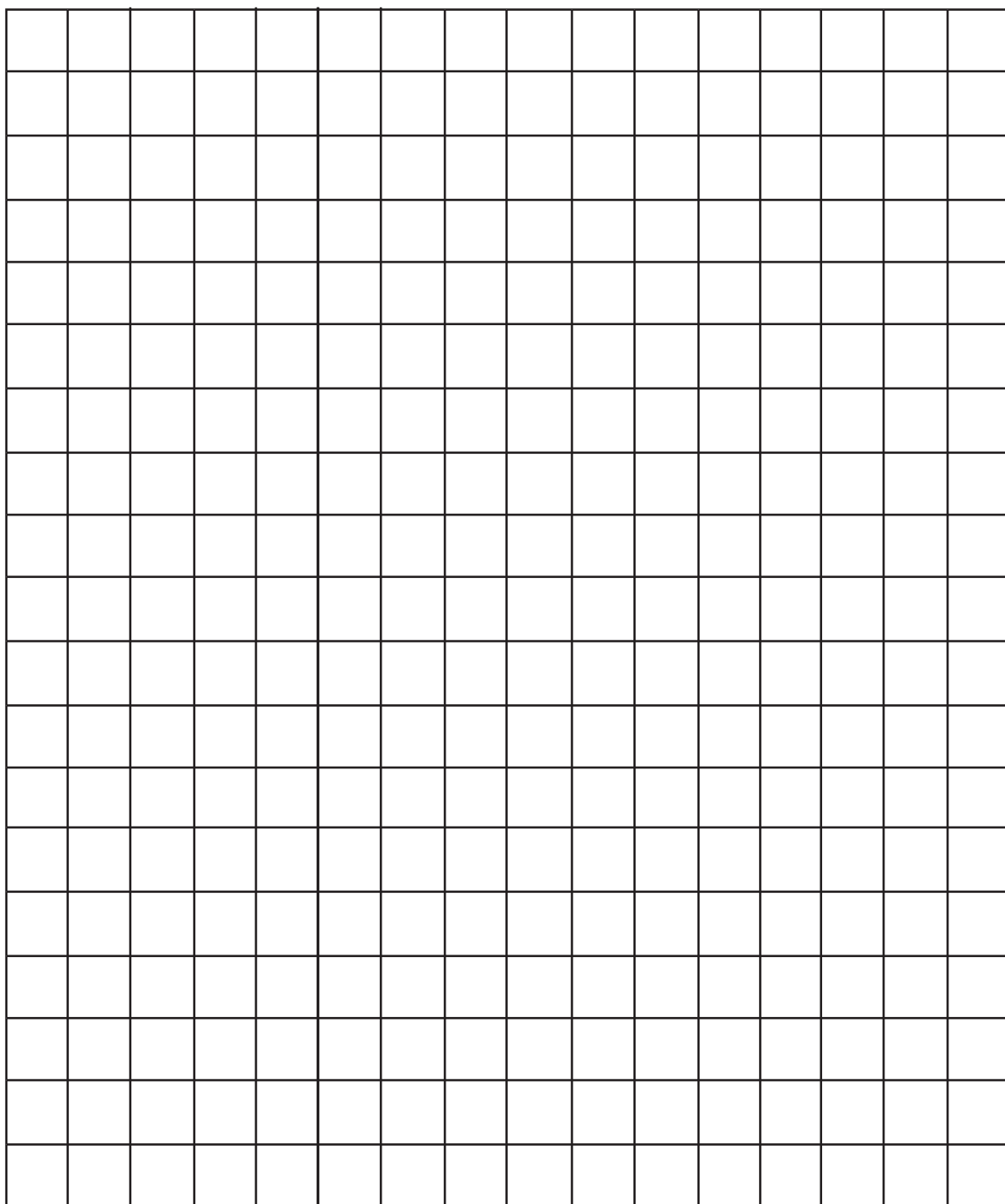


Place-Value Comparison

Whole-Number Place Value	Decimal Place Value



Centimeter Grid Paper





Perfect Squares

Numbers such as 4, 9, 16, and 25 are called *perfect squares*.

Perfect squares have a factor pair that are identical factors.



Square Roots

Perfect Squares \rightarrow

$$\begin{aligned}4 &= 2^2 \\9 &= 3^2 \\16 &= 4^2\end{aligned}$$

Radical Symbol $\sqrt{\quad}$

Asking for the repeated factor that created the product (area).

For a square, radical symbols ask for the side length of the square.

Square Root

$$\sqrt{4} = 2$$

$$\sqrt{9} = 3$$

$$\sqrt{16} = 4$$

Rational Numbers

Whole number answers or fractional answers are referred to as *rational numbers*.



Irrational Numbers

Irrational numbers are non-terminating (never ending), non-repeating decimals.

For roots, it tells us there is no exact identical factor for the side of the square to create the given area.



Rational and Irrational Numbers



Rational Numbers

(those that are whole or fractional)

Provide an exact answer that may be easily placed
on a number line or drawn

Irrational Numbers

Always approximations because the decimal
never ends



Estimating Roots

Try estimating $\sqrt{5}$

- To approximate a root, find what two other perfect squares the number falls between.
- It was shown that $\sqrt{4} = 2$.
- The next perfect square is 9; so $\sqrt{9} = 3$.
- The $\sqrt{5}$ is between $\sqrt{4}$ and $\sqrt{9}$, but must be closer to $\sqrt{4}$.
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Module C

Instructor's Guide



Module C: Number Sense and Fractional Concepts

A. Module Goals

Use the **Module Goals** transparency (T1) to review the module goals.

Module C: Number Sense and Fractional Concepts

The paraeducator will:

1. Use number sense to justify the reasonableness of solutions for a variety of computation and problem-solving situations
2. Use concrete materials to develop fractional concepts for addition and subtraction
3. Use concrete materials to develop fractional concepts of multiplication and division



Goal 1: *Use number sense to justify the reasonableness of solutions for a variety of computation and problem-solving situations.*



1.1 Lecture: Number Sense with Fractions, Decimals, and Percentages



***Note to Instructor:** Remind participants to use their math journals to take notes during lecture periods.

A common problem with mathematical development is the focus on algorithms over understanding. Students who commonly struggle with new concepts such as fractions have little true understanding of the actual concept. Those with a deeper understanding are able to plan and evaluate a problem-solving situation and find more immediate success.

The next three modules in this Academy focus on fractions, decimals, and percentages. These concepts are pivotal for many students in determining their future attitude towards and success in mathematics.

Before approaching each concept in depth, it is important to identify and evaluate knowledge that students bring from early elementary grades.

One concept that often gets overlooked is the development of *number sense*. Use the **Number Sense** transparency (T2).

Number Sense

The ability to evaluate whether an answer is reasonable and appropriate.

Number sense also implies that a student should be able to employ a variety of strategies to make a prediction. This is not a skill that is easily taught. Through practice, students should be able to predict the final outcome and use computation strategies as verification.



Estimation strategies are vital to developing number sense. For fractions and decimals, students should be able to project an outcome by rounding to the nearest whole or half. Fractions present more of a difficulty when odd *denominators* are present.

The following activity includes a variety of strategies to help decide on the appropriateness of an answer. Allow participants to share their logic in reasoning. This helps develop new strategies for other classmates.



1.2 Activity: Make Your Best Guess

The paraeducator will use prior estimation strategies to justify the reasonableness of an answer.

Materials:

- **Make Your Best Guess** handout (H1)



1.2.1 Steps

- Set ground rules of not using calculators and algorithms (do not solve the actual problem).
- Many of these problems have complex algorithms for computation. However, that is not the point of this activity, so no algorithms are needed.
- Suggest that participants think of making the numbers easier to work with mentally.
- Have participants complete the handout individually. Make sure they write what strategy they used to decide if the answer was appropriate.
- Answers: Explanations will vary
 - ▲ 1. $2000 \rightarrow$ must be larger than 1000 as there is already a number over 1000 and the problem is adding.
 - ▲ 2. $1 \rightarrow 5.0076$ is close to 5 and 3.978 is close to 4, so $5 - 4 = 1$.
 - ▲ 3. $4 \rightarrow 1.95$ and 2.01 are close to 2, so the product must be near 4.
 - ▲ 4. $4 \rightarrow 1\frac{1}{6}$ is close to 1 because the fraction is near zero; $2\frac{7}{8}$ is close to 3 because the fraction is near one whole; $\frac{3}{11}$ is close to zero so will not have an affect; the total is near 4.
 - ▲ 5. $8 \rightarrow 2\frac{1}{5}$ is close to 2 because the fraction is near zero; $3\frac{8}{9}$ is near 4 because the fraction is near one whole; the product is 8.
- Discuss ideas as a class. Participants should note that each problem may be changed into something that is easily manipulated mentally.
- If students are unable to judge the size of a fraction, introduce more concrete activities such as fraction bars before any use of computation rules can begin.
- Teachers are role models for problem solving skills such as estimation. Estimation is a skill that often eludes students because they cannot “see” the process that is occurring. It is important that teachers communicate their thought processes to help students develop mathematically.



Goal 2: Use concrete materials to develop fractional concepts for addition and subtraction.



2.1 Discussion: Adding and Subtracting Fractions with Common (Like) Denominators

The paraeducator will use concrete experiences to create rules for adding and subtracting with common (like) denominators.

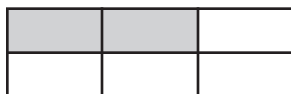
Materials:

- **Centimeter Grid Paper** handout (H2)
- **Adding Fractions** transparency (T3)

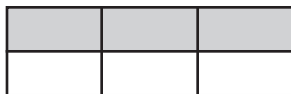


2.1.1 Steps

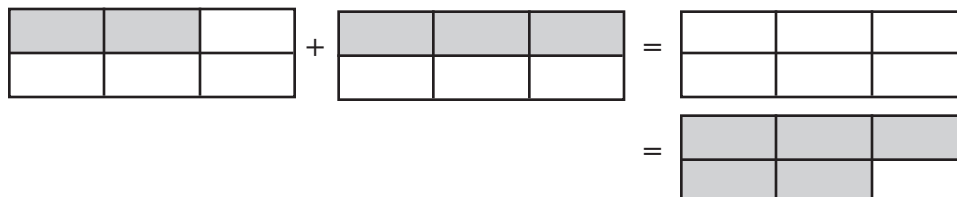
- Review the terms *numerator* (upper part of the fraction – represents the part or chosen pieces) and *denominator* (bottom part of the fraction – represents total parts in the whole).
- The most common error with fractions involves the role of the denominator.
- Place the **Adding Fractions** transparency on the overhead (T3) and ask the participants to name the fraction of shaded rectangles (2/6).



- Ask the following questions to confirm participants' basic knowledge:
 - ▲ What is the total number of parts in whole? (6)
 - ▲ How many parts are shaded? (2)
 - ▲ Why is the fraction not 2/4? (This would imply 2 out of 4 total parts – did not ask for shaded vs. unshaded)
- Uncover the next picture and follow the above questions. (3/6)



- Reiterate that in each picture, there are six parts in the whole.
- Place an addition sign between the two pictures. Show the shading.
- Explain that we would like to add the two groups of shaded portions.
 - ▲ Discuss that many students state the answer as 5/12.
 - ▲ They see combining diagrams as now 12 parts to the whole.
 - ▲ Students stating 12 parts as the whole do not understand that the whole has not changed – it still has six equal parts.





- Rephrase and clarify the concept:
 - ▲ How many total shaded parts? (5)
 - ▲ When we add, we have to fill in any empty spaces on a single model. First shade two rectangles. Then shade three more.
 - ▲ The whole for each model has six parts, so the final answer must also have six parts.
 - ▲ The answer is $\frac{5}{6}$.
 - ▲ Make note that there are still only six total pieces to the whole.
- Ask participants to use the centimeter grid paper (**H2**) to demonstrate (draw) the following problems. Include the actual fraction work under each picture.
 - ▲ $\frac{2}{5} + \frac{1}{5} =$
 - ▲ $\frac{3}{4} + \frac{2}{4} =$
- Note that there can be any number of drawings, including a straight line of squares or some shape with the correct total parts. We use rectangles for ease of drawing. Students need exposure to a variety of fractional spaces (this includes circles).
 - ▲ $\frac{2}{5} + \frac{1}{5} = (\frac{3}{5})$
 - This is similar to the example above.
 - ▲ $\frac{3}{4} + \frac{2}{4} = (\frac{5}{4} = 1\frac{1}{4})$
 - This presents a new challenge. The whole has four parts of which all get filled in.
 - The leftover piece must start another whole of four. Some students then make the mistake of saying there are eight total pieces.
 - If a student gives the answer of $\frac{5}{8}$, this would imply that a little more than one half of one whole was covered. Using number sense, $\frac{3}{4}$ is already over a half and $\frac{2}{4}$ is a half, so our answer must equal at least one whole; thus, $\frac{5}{8}$ cannot be the answer.
 - Reiterate the whole concept (in this case four parts is a whole).
 - We can also write the answer as $1\frac{1}{4}$, which implies one whole and one quarter of another whole was shaded.
 - *Note:* Fractions such as $1\frac{1}{4}$ are called mixed numbers because they include a whole number and a fraction. Fractions such as $\frac{5}{4}$ are called improper fractions because the numerator is larger than the denominator. These will be covered in depth in the next activities but may be introduced here if appropriate.
- Ask participants to provide a rule to explain how to add fractions with like denominators. Record the rule on the transparency **Adding Fractions (T3)** and make sure the paraeducators record it in their journals. Possible steps include:
 - ▲ Check the denominators – must have same number of parts in the whole
 - ▲ Add the numerators (number of parts)
 - ▲ Place the new numerator over the common denominator
- Ask participants to draw and discuss the following. Use centimeter grid paper.
 - ▲ $\frac{5}{8} - \frac{2}{8} =$
 - ▲ Ask for methods of showing the process with a drawing



- Remind participants that addition and subtraction are inverse processes which implies that we start with the total.

- Students can “x” or cross out 2/8.

x			
x			

- This leaves $3/8$.
- ▲ Students do not usually make the same denominator error that they do with addition.
- Ask participants to draw the following. Use centimeter grid paper.
 - ▲ $13/6 - 4/6 =$
 - ▲ Getting started in this problem is the most difficult because of $13/6$ (improper fraction)
 - ▲ Ask participants what they know about
 - Greater than one
 - Each whole has six parts
 - One whole is $6/6$ shaded
 - ▲ Must start by drawing (improper fraction)

- ▲ Note that we could name this as $2-1/6$ (mixed fraction)
- ▲ By crossing out the $4/6$, that leaves $9/6$ (or $1-3/6 = 1-1/2$)
- As a group, create a rule for subtracting fractions with like denominators. Use the **Subtracting Fractions with Like Denominators** transparency (T14). Record in journals. Possible steps include:
 - ▲ Check the denominators – must have same number of parts in the whole
 - ▲ Subtract the numerators (number of parts)
 - ▲ Place over the common denominator



2.2 Activity: Fraction Friction

The paraeducator will use concrete materials to create rules for adding and subtracting fractions with unlike denominators.

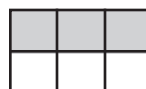
Materials:

- Pattern blocks
- Overhead pattern blocks
- Transparency/handout **Fraction Friction (T5/H3)**
- Transparency **Unlike Denominators (T6)**

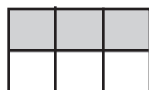
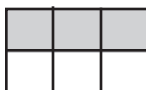


2.2.1 Steps

- Ask each participant to get out 1 blue rhombus, 1 yellow hexagon, 1 green triangle, and 1 red trapezoid.
- Explain that the hexagon will be our whole.
- As a group, define the fractional value for each of the other blocks.
 - ▲ As a hint, figure out how many of each color it would take to cover the hexagon.
 - ▲ Trapezoid: two trapezoids cover the hexagon so one trapezoid is $1/2$.
 - ▲ Rhombus: three rhombuses cover the hexagon so one rhombus is $1/3$.
 - ▲ Equilateral triangle: six triangles cover the hexagon so one triangle is $1/6$.
- Give the problem $1/2 + 3/6 =$
 - ▲ Encourage the participants to attempt to draw this problem on their centimeter grid paper. Use the transparency **Unlike Denominators (T6)**.
 - ▲ Ask them to identify potential mistakes that a student might make when solving this problem.
 - The answer is not $4/8$
 - Cannot combine as written because of unlike denominators (not same total parts)



- ▲ Need to make common pieces. For this example, we can subdivide our drawing by adding more lines to the $1/2$ to make it into six parts. Show new subdivisions on the overhead diagram **(T6)**.



- ▲ Subdividing drawings can be difficult because the student must organize the shading; otherwise, the equivalence is not obvious.
- ▲ When we add these together, as with common denominators, we get $3/6 + 3/6 = 6/6 = 1$.
- Not all problems have drawings that are simple to subdivide because the denominators do not have an obvious relationship such as $1/2 + 1/3 =$.
- Return to the first example of $1/2 + 3/6 =$ and present another spatial model using the pattern blocks. Use overhead pattern blocks to demonstrate the same problem. Use transparency/handout **Fraction Friction (T5/H3)** and fill in the handout.
 - ▲ Have participants choose the piece that represents $1/2$ (one trapezoid).
 - ▲ Have participants choose a way to represent $3/6$.
 - As there is no one piece, we must determine how to make $3/6$ (three triangles)
 - ▲ Participants can visually see that we have different pieces so the fractions cannot be added together as written (no common denominator).
 - ▲ Look for “trade-ins” that would make it easy to add. Use the transparency/handout **Fraction Friction** to guide the experience for #1.



- We can trade the trapezoid for three green triangles to make a common piece
- $1/2 + 3/6 = 1$ trapezoid + 3 triangles = 3 triangles + 3 triangles = 6 triangles = $6/6 = 1$ hexagon = 1
- Mathematically, $1/2 + 3/6 = 3/6 + 3/6 = 6/6 = 1$
- ▲ Ask the group to figure out the second method for #1.
 - For this example, the $3/6$ (3 green triangles) may be traded in for 1 trapezoid ($1/2$), giving $1/2 + 1/2 = 2/2 = 1$
 - $1/2 + 3/6 = 1$ trapezoid + 1 trapezoid = 2 trapezoids = $2/2 = 1$ hexagon = 1
- ▲ Return to $1/2 + 1/3$. Demonstrate on the overhead for #2 on the **Fraction Friction** handout/transparency.
 - Have participants create each fraction with their blocks ($1/2 = 1$ trapezoid and $1/3 = 1$ rhombus) and record the starting problem on **Fraction Friction** handout/transparency (**H3/T5**) (same as modeled in #1): 1 trapezoid + 1 rhombus
 - They cannot be combined as written because they are different blocks (different denominators)
 - A rectangular diagram would not show the necessary divisions easily – we will use the pattern blocks
 - Discuss what piece is “common” to both shapes (green triangles) – this can be difficult for students to see because the triangles are not present in the original problem
 - Ask the participants to “trade in” pieces and create common pieces covering the same area
 - Record answers for #2 (second step): 3 triangles + 2 triangles = 5 triangles or $5/6$
 - Complete the problem mathematically: $1/2 + 1/3 = 3/6 + 2/6 = 5/6$
- ▲ Discuss where each number (numerator and denominator) is found within this model
 - Numerator → total number of common pieces
 - Denominator → type of piece
- Ask participants to complete the **Fraction Friction** handout/transparency (**H3/T5**) with the pattern blocks in pairs. Make sure that they describe the original pieces, the new pieces, and the mathematical final answer.
 - ▲ $1/3 + 1/6 = 1$ rhombus + 1 triangle = 2 triangles + 1 triangle = 3 triangles = $3/6$.
 - For the last example, if triangles are used, we always present the final answer in its simplest form (or fewest pieces).
 - $1/3 + 1/6 = 3/6$ can be traded in equally for a single trapezoid or $3/6 = 1/2$.
 - ▲ $1/2 - 1/3 = \text{trap.} - 1 \text{ rhombus} = 3 \text{ triangles} - 2 \text{ triangles} = 1 \text{ triangle} = 1/6$.
 - ▲ $1 - 1/3 = 1 \text{ hexagon} - 1 \text{ rhombus} = 6 \text{ triangles} - 2 \text{ triangles} = 4/6 = 2/3 = 3 \text{ rhombuses} - 1 \text{ rhombus} = 2 \text{ rhombuses} = 2/3$.

- four triangles can be traded in equally for two rhombuses, which is equal to the second equivalent expression.
- Using the **Fraction Friction Reflection** transparency (T8), have participants share and record their reflections about patterns they see and procedures that could be shared with students from this experience. Possible answers include:
 - ▲ The common piece is often smaller than the given pieces.
 - ▲ When the denominator changes, the numerator also changes.
 - ▲ When a smaller piece is used, the numerator gets larger.
 - ▲ A larger number in the denominator means a smaller piece (or more were required because they were smaller).

Note: This concept is particularly challenging for students to understand. A larger denominator means a smaller piece.

 - ▲ The denominator is a multiple of the two original denominators (this is very important).
- For their journals suggest that participants reflect on issues that students will encounter using these methods (shading and pattern blocks).
 - ▲ Without graph paper, students will not draw equivalent figures to shade
 - ▲ Students may not shade the diagram to see equivalent areas
 - ▲ Pattern blocks are limited to certain denominators
 - ▲ Difficult to carry over to traditional algorithm



2.3 Activity: All Mixed Up

The paraeducator will develop an algorithm for changing improper fractions to mixed fractions and mixed fractions to improper fractions from concrete experiences.

Materials:

- **Types of Fractions** handout and transparency (H5/T9)
- **All Mixed Up** handout (H6)
- **Centimeter Grid Paper** handout (H2)



2.3.1 Steps

- Put the following problem on the overhead (see **Types of Fractions** handout and transparency).



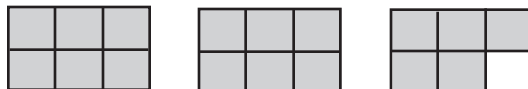
- ▲ This is a typical fraction known as a *proper fraction*.

Proper fraction: A fraction whose numerator is smaller than the denominator.

- ▲ Name the fraction on the handout ($5/6$).



- Use the following model for both mixed and improper fractions. Ask participants to name the fraction.



- ▲ From the prior activity, participants will likely name this as $17/6$. This is an *improper fraction*.
- ▲ Ask participants to verbally define *improper fractions* based on prior knowledge.

Improper fraction: A fraction whose numerator is larger than the denominator.

- ▲ Ask participants to name this picture in a different way. This may be named as $2\text{-}5/6$. This is called a *mixed fraction*.

Mixed fraction: A fraction that has both a whole and a fractional part.

- Add another key component to the definition of *improper* and *mixed fraction* definitions:
 - ▲ Both *improper* and *mixed fractions* imply larger than one whole.
 - ▲ This is a key part of developing a fraction sense.
- Have participants complete the handout **All Mixed Up (H6)** in pairs with only this information. Do not check answers at this point.

Discussion following the activity:

- Participants completed the activity from visual pictures. This may be a long procedure if there is no diagram or the denominators are difficult. An algorithm is necessary to make the work more efficient.
- Start with the diagram for the mixed number $2\text{-}5/6$ from the **Types of Fractions** transparency (T9).
 - ▲ Ask participants to see if they can determine how to use all of the numbers present and end up with the equivalent improper form $17/6$.
 - ▲ Participants should find the following for the initial example:
 - The denominator did not change
 - Using $6 \cdot 2 + 5 = 17$ creates the numerator
- Have participants draw $1\text{-}1/2$ on their centimeter grid paper as a reference. Name as the improper equivalent. Test all group-suggested rules for changing from mixed to improper on $1\text{-}1/2$ (should equal $3/2$) to confirm or refute.





- Create a rule for converting *mixed fractions to improper fractions*. Add to the **Types of Fractions** handout and transparency.
 - ▲ *Converting mixed fractions to improper fractions*: Multiply the denominator by the whole number and add the numerator. Place over the same denominator
- Discuss this rule conceptually for $2\text{-}5/6 = 17/6$:
 - ▲ The $6 \cdot 2$ portion shows two groups of six parts (return to the original diagram)
 - ▲ Adding 5 accounts for the leftover shaded pieces
 - ▲ Placing the total pieces (17) over 6 shows that there were six pieces per whole and 17 sixths were shaded in
- Have participants put into words $1\text{-}1/2 = 3/2$:
 - ▲ The $2 \cdot 1$ portion shows one group of two parts (return to the original diagram)
 - ▲ Adding 1 accounts for the leftover shaded pieces
 - ▲ Placing the total pieces (three) over 2 shows that there were two pieces per whole and three halves were shaded in
- From the above discussion of the rule, challenge participants to start with $17/6$ and return to $2\text{-}5/6$.
- Test all suggested rules as before. Participants should find the following:
 - ▲ Going in the opposite direction means reversing all the operations in the above rule.
 - ▲ The denominator did not change
 - ▲ Using basic division concepts identifies how many whole groups and partial groups may be created from the total
 - ▲ The denominator tells how many shaded pieces in each group (dividing by 6). The quotient (2) tells how many whole groups. The remainder (5) tells how many shaded parts out of six were left

$$\begin{array}{r} 2 \\ 6 \overline{) 17} \\ - 12 \\ \hline 5 \end{array}$$

- ▲ The above problem shows $2\text{-}5/6$
- Create a rule for converting *improper fractions to mixed fractions*. Add to the **Types of Fractions** handout.
 - ▲ *Converting improper fractions to mixed fractions*: Divide the denominator into the numerator. The quotient (answer) is the whole number. The remainder is the new numerator for the fractional part over the original denominator.



2.4 Activity: Continued

- Have participants return to the **All Mixed Up** handout (**H6**) to use the new rules to check their work.
- Answers for the **Mixed Up** handout:
 - ▲ $3\text{-}1/2 = 7/2$ (circle diagrams)
 - ▲ $2\text{-}1/3 = 7/3$ (rectangle into thirds)
 - ▲ $1\text{-}2/5 = 7/5$ (long rectangle)
 - ▲ $2/8 = 1/4$



2.5 Discussion and Lecture: Common Algorithms for Adding and Subtracting Fractions

The prior activity provided a concrete experience on which students can base further algorithmic work. It is vital to link algorithmic steps to concrete experiences to further a deeper understanding. Included here is a brief explanation of the key ideas for fraction manipulation in addition and subtraction.

Adding and Subtracting with Like Denominators

Adding and subtracting fractions with like denominators only involves the numerators as shown in the above discussions and activities. Fraction bars may be used to move from the prior drawings to an algorithm. Use the handout and transparency **Fraction Bars (H7/T10)** to work with the algorithm. In a classroom, students would have their own sets of fraction bars to work the problem. These can be time consuming to make but are very useful when students need extra assistance at home.



***Note to Instructor:** If scissors are available, participants may want to cut out the whole bars (do not cut apart individual fractional pieces). They can fold back the bars to show a particular length.

Show an easy problem on the overhead: $1/5 + 2/5$. Ask participants how they might use fraction bars to show this problem. They should suggest tracing the bar $1/5$ long and adding on the tracing of the bar $2/5$ long. To find the answer, line up the “total” length against the other bars. They should see that $1/5 + 2/5$ is the same length as $3/5$. So $1/5 + 2/5 = 3/5$.

Demonstrate the subtraction concept by starting with a longer bar such as $3/4$ and cross off a length of $2/4$ to show the problem $3/4 - 2/4$. The length left can be matched to the bar that is $1/4$ long. So $3/4 - 2/4 = 1/4$. Again, the algorithm for adding and subtracting like denominators is to add or subtract the numerators and place the sum over the common denominator.

Adding and Subtracting Fractions with Unlike Denominators

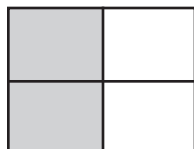
Use the **Adding Fractions with Unlike Denominators** handout and transparency **(H8/T11)**. The algorithm transition for adding and subtracting unlike denominators is more difficult. Return to the fraction bar concept for reference. Use $1/3 + 1/4$ and trace the bars as before. Participants should notice that the total length is now $7/12$ by comparing it to the twelfths bar. The fraction sentence is $1/3 + 1/4 = 7/12$. This is a difficult concept to accept mathematically. In the earlier activity with pattern blocks, we noticed that the number of parts as well as the type of piece changed. This problem suggests that we must somehow change the type of “piece,” or the denominator, in order to add the fractions.

Number theory concepts are necessary to solve our problem. Ask participants what relationship they know exists between the numbers in the denominators (3, 4, and 12). They

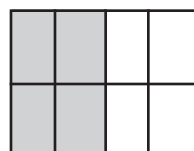


should see that 3 and 4 are factors of 12; 12 is a multiple of 3 and 4. This relates back to the concept of *least common multiple (LCM)*. A common denominator is formed from the LCM. Once a common denominator is found, we must create an *equivalent* or *equal fraction*.

To reiterate *equivalent fractions* use the transparency **Equivalent Fractions (T12)**. Use a scrap piece of paper and fold it in half. Shade half of the paper. Participants should name this as $1/2$. Fold the paper again to make four parts.



Ask participants to name the shaded area. They should be able to name $2/4$. This means that $1/2 = 2/4$ (equivalent fractions). Add to the transparency. Ask participants what the next equivalent fraction would be if we create eights parts (fold it back into fourths and fold again to create eight parts). The next equivalent fraction is $4/8$. Add another equal statement as shown below.



$$1/2 = 2/4 = 4/8$$

Ask participants to explain mathematically how $2/4$ would be created from $1/2$. They should see that both the numerator and the denominator were multiplied by 2. The key to equivalent fractions is multiplying the numerator and the denominator by the same factor. Fraction bars can be used once again to see more examples that are the same length but written differently by choosing a length and scanning down the sheet: $1/3 = 2/6 = 3/9$.

Returning to $1/3 + 1/4 = 7/12$ on **Adding Fractions with Unlike Denominators** transcript/handout (T11/H8), 12 is our common denominator, and equivalent fractions must be created to be able to add the fractions.

$$\begin{array}{r} 1/3 = \quad /12 \\ + \quad 1/4 = \quad /12 \\ \hline \end{array}$$

From here, each missing factor must be found to make the new denominator. For the first fraction, 3 must be multiplied by 4 to get 12. This means that the numerator must also be multiplied by 4.

$$\frac{1 \cdot 4}{3 \cdot 4} = \frac{\quad}{12}$$



This shows that $4/12$ is the equivalent fraction of $1/3$. Check on the fraction bars sheet. The second fraction is missing the factor of 3.

$$\frac{1 \cdot 3}{4 \cdot 3} = \frac{\quad}{12}$$

This shows that $3/12$ is the equivalent fraction of $1/4$. Check on the fraction bars sheet.

The new problem is $4/12 + 3/12$. Using the rules for like denominators, $4/12 + 3/12 = 7/12$.

The same process may be used for subtraction. Using the transparency/handout **Subtracting Fractions with Unlike Denominators (T4/H9)**, complete the steps on the transparency/handout. For $2/3 - 1/6$, a common denominator is needed. Using our LCM knowledge, 6 is the common denominator. Equivalent fractions must be formed.

$$\begin{array}{r} 2/3 = \quad /6 \\ - 1/6 = \quad /6 \\ \hline \end{array}$$

For this example, the second fraction is already completed. Use the fraction bars to confirm.

$$\begin{array}{r} \frac{2 \cdot 2}{3 \cdot 2} = \frac{4}{6} \\ - 1/6 = 1/6 \\ \hline 3/6 \end{array}$$

$$2/3 - 1/6 = 3/6$$

Any problem involving mixed numbers may be simplified by changing any mixed number to an improper fraction and following the above processes. While this may not be the method taught in many classrooms, it is a student-friendly way to solve these difficult problems. Consultation with the classroom teacher is important for this concept. Show the following problem on a blank transparency.

$$\begin{array}{r} 1-3/4 = 7/4 = 21/12 \\ + 2-1/3 = 7/3 = 28/12 \\ \hline \end{array}$$

This gives the solution of $49/12$. Students should use number sense to rationalize this solution. $1-3/4$ is close to 2 and $2-1/3$ is close to 2, so the answer should be around 4. Our answer checks with our estimation.

Simplifying Fractions

In an earlier problem, the solution was $3/6$. While correct, $3/6$ is not in the *simplest* or *reduced form*. Use the transparency and handout **Simplifying Fractions (T14/H10)**.



Simplest Form

The numerator and denominator have no common factor except 1.

With the pattern blocks, smaller blocks were traded in for fewer equivalent blocks to put the fraction in simplest form. You may wish to return to the **Fraction Friction (H3)** to review this point.

While the LCM was used to create equivalent fractions, the *greatest common factor (GCF)* is used to reduce a fraction. This is a progressive concept starting with a student's knowledge of division and multiplication. Initially, students are taught to "find the largest number that will divide the numerator and denominator evenly" in order to reduce a fraction. This means trial and error. For example, on the handout **All Mixed Up (H3)**, participants found that $2/8$ was equivalent to $1/4$. The fraction $2/8$ is not in simplest form. Students are asked to find the largest number (factor) that 2 and 8 have in common. Students should easily guess that the GCF is 2, from their knowledge of even numbers. The fraction $2/8$ may be reduced by $(2 \div 2)/(8 \div 2) = 1/4$. We know that $1/4$ is in simplest form as 1 and 4 have no factors in common except 1.

For more difficult fractions students may need to reduce twice if they do not start with the GCF. Continue using transparency/handout **Simplifying Fractions** to show different methods of reducing. For example, $12/36$ may be reduced by many numbers. If we start with the factor of 4, then $(12 \div 4)/(36 \div 4) = 3/9$. While this is a correct place to start, $3/9$ is not in the simplest form because the numerator and denominator share a common factor of 3. Reducing further, $(3 \div 3)/(9 \div 3) = 1/3$. The actual greatest common factor was 12 (noting the division by 4 and then a division by 3). Show how 12 reduced the fraction in one step. To help students with larger fractions, a grade-level appropriate strategy would be to use the number theory concept of *prime* numbers.

Use either method (trees or birthday cake) to find the prime factorizations of the numerator and denominator in $12/36$. The result should be $12/36 = 2 \cdot 2 \cdot 3 / 2 \cdot 2 \cdot 3 \cdot 3$. A fraction reduces by canceling out common factors in the numerator and denominators. Each number has two 2's and one 3. Continue using the transparency/handout

Simplifying Fractions.

$$\frac{12}{36} = \frac{\cancel{2} \cdot \cancel{2} \cdot \cancel{3}}{\cancel{2} \cdot \cancel{2} \cdot \cancel{3} \cdot 3}$$

Knowledge of basic division must be used to understand the numerator. Many students will say that zero is left in the numerator. This shows a lack of understanding in basic division. When dividing by the same number, the quotient is 1. The problem becomes

$$\frac{12}{36} = \frac{1 \cdot 1 \cdot 1}{1 \cdot 1 \cdot 1 \cdot 3} = \frac{1}{3}$$

This method, while at a higher level of understanding, reduced the fraction in one step. From the prior method, 12 was found to be the GCF. This method supports that conclusion as $2 \cdot 2 \cdot 3 = 12$.



Goal 3: Use concrete materials to develop fractional concepts of multiplication and division.



3.1 Discussion: Understanding Multiplication and Division of Fractions

The paraeducator will represent the concept of multiplying and dividing fractions using sentences.

Materials:

- **Multiplication and Division Concepts** handout and transparency (H11/T15)
- White blank scrap paper
- Blank transparencies
- Contrasting colors of markers



*** Note to Instructor:** Throughout this goal, several spatial models are introduced. Not every participant will like every model. Many will just want the algorithm. However, it is important that participants see the progression and links between these models and the algorithm. Paraeducators must recognize that various methods are necessary to teach children these concepts. Provide additional examples as needed to make sure that participants grasp the concept. See the resources page at the end of this Academy for additional web practice on these concepts.

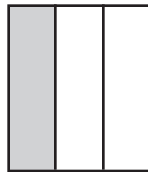


3.1.1 Steps

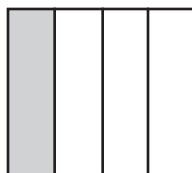
- The mathematical process for multiplication and division of fractions is simpler than for addition and subtraction of fractions; however, the conceptual understanding is more difficult.
- Use the transparency/handout **Multiplication and Division Concepts (T15/H11)** to start with a review of basic whole-number multiplication concepts.
 - ▲ Using the problem $3 \cdot 4$ (#1), record participants' responses on the transparency and handout for related addition sentences
 - $3 + 3 + 3 + 3$ or $4 + 4 + 4$
 - ▲ Have participants share their verbal interpretation (#2) of $3 \cdot 4$ as multiplication. Record on transparency and handout
 - three groups of 4 or four groups of 3
 - both interpretations produce the answer of 12
- Fraction multiplication may be developed from similar concepts.
- Ask participants to verbally and mathematically interpret the problem $5 \cdot \frac{2}{3}$; use the handout and transparency **Multiplication and Division Concepts (H11/T15)**.
 - ▲ Verbal interpretation: five groups of $\frac{2}{3}$ (#3)
 - ▲ Mathematical interpretation: $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3}$ (#3)
 - ▲ Encourage participants to use their knowledge of adding fractions with like denominators to solve the addition problem ($\frac{10}{3} = 3\frac{1}{3}$) (#4)
- The problem $\frac{2}{3} \cdot 6$ is challenging to represent verbally. The phrase “two-thirds groups of 6” doesn't really make sense.



- ▲ One important point to note for this problem is that order does not matter in multiplication (due to the commutative property of multiplication) so the problem may be restated as $6 \cdot \frac{2}{3}$, which aligns with the first example.
 - Verbal interpretation: 6 groups of $\frac{2}{3}$ (#5)
 - Mathematical interpretation: $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3}$ (#5)
- ▲ A new verbal interpretation of fraction multiplication may start with the same problem. The original problem, $\frac{2}{3} \cdot 6$, may be restated as “two-thirds *of* six.” The word *of* in mathematics means to multiply.
- Start with a simpler version of this problem and use the rectangle shading model: $\frac{1}{3}$ *of* 6. This problem implies shading one third *each* of six wholes. Show one diagram as a reminder (show on a blank transparency).



- For six wholes, the related addition problem is $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{6}{3} = 2$ wholes.
- The initial problem asked for $\frac{2}{3}$, which is twice the fraction $\frac{1}{3}$. So the related problem asks for twice the result of $\frac{1}{3}$ *of* 6, which would be four wholes. Then $\frac{1}{3} \cdot 6 = \frac{12}{3} = 4$.
 - ▲ This problem may have been solved using the initial $\frac{2}{3}$ written six times as demonstrated above
- The most difficult piece to understand is a “fraction of a fraction” such as $\frac{2}{3} \cdot \frac{1}{4}$ (#6).
- Ask participants to put this problem into words (#6).
 - ▲ Two-thirds *of* one fourth
 - ▲ This is a fraction of another fraction
- Use a blank piece of scrap paper to demonstrate the concept (#6).
 - ▲ Start with one fourth because the problem states “of one fourth” (*note that the problem could start with two-thirds if the order was changed*)
 - ▲ Fold the piece into fourths using non-intersecting folds and shade one fourth as done in prior activities



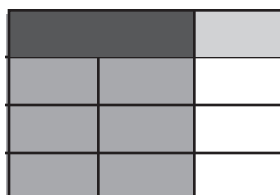
- ▲ The problem asks for two-thirds *of* that shaded area
- ▲ Turn the paper to the other orientation



- ▲ Ignoring the prior folding and shading, fold the paper into thirds (denominator) and shade two thirds in a contrasting color



- ▲ Note that there are two rectangles with both colors. The overlap is the answer to the problem. In order to take a fraction of a fraction, the total amount of pieces must be readjusted into smaller pieces



- ▲ The final answer for $2/3 \cdot 1/4 = 2/12$
- Ask participants if they can see how the final product was created mathematically.
 - ▲ Multiplying the numerators and then the denominators
- Have participants finish #7 and #8 on the handout.

Division Concepts

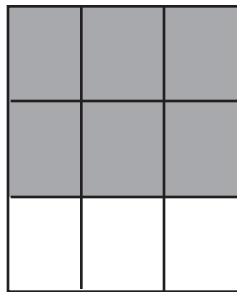
- Division will be treated similarly to multiplication starting with common division concepts on the **Multiplication and Division Concepts** handout and transcript (**H11/T15**).
- Ask participants to verbally interpret the division problem (#9).
 - ▲ Takes 12 and divides it into three equal groups
 - ▲ Takes 12 and divides it into groups of three
 - ▲ Repeated subtraction: $12 \div 3 = 12 - 3 - 3 - 3 - 3$
- Fraction division can be developed from similar concepts.
- Ask participants to verbally interpret the problem $1 \div 1/2$ (handout and transparency **Multiplication and Division Concepts**). Make sure they have the following. These are difficult without guidance.
 - ▲ Verbal interpretation: 1 divided into groups of $1/2$ (#10)
Note: It is confusing to say 1 divided into $1/2$ groups although it actually implies the same idea.
 - ▲ It is helpful to phrase this as a question: *How many groups of $1/2$ are in 1?*
- To solve, this may be written using repeated subtraction (How many times can $1/2$ be subtracted from 1?).
 - ▲ Repeated subtraction: $1 - 1/2 - 1/2$ (subtracted twice before going to zero) (#10)
 - ▲ The solution is 2 groups or $1 \div 1/2 = 2$



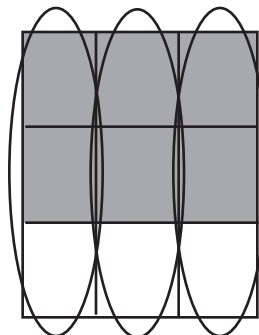
- The problem $2/3 \div 3$ is difficult to describe. Verbally it asks, “If two-thirds is divided into three equal groups, how many shaded pieces are there in each group?” This is difficult to understand because it seems to ask about wholes that do not exist because two-thirds is less than 1 (#11).
- Use another scrap piece of paper and fold and shade $2/3$.



- ▲ Turn the paper around and create three sections.



- ▲ Nothing is additionally shaded this time as the question is asking, within each group, how many pieces are shaded. It helps to return to the relationship between addition and subtraction used earlier as subtraction starts with the total.
- ▲ Thinking in terms of a pizza, the question may be asked as follows:
 - Suppose you had $2/3$ of a pizza and you wish to share it equally among three friends. How much of the pizza would each person receive?



- ▲ In each group, there are two of nine total pieces that are shaded. So, $2/3 \div 3 = 2/9$.



- Fraction division by another fraction will be covered in the following activity with the pattern blocks. Conceptually it works similarly to the previous example.



3.2 Activity: Fraction Friction 2

The paraeducator will use concrete materials to further understanding of multiplication and division of fractions.

Materials:

- Pattern blocks
- **Fraction Friction 2** handout and transparency (H4/T7)



3.2.1 Steps

- Returning to the pattern blocks from earlier activities, the whole is now defined as two hexagons. Ask participants to redefine each piece (think: How many of each piece does it take to cover the two hexagons?).
 - ▲ 1 yellow hexagon = $1/2$
 - ▲ 1 red trapezoid = $1/4$
 - ▲ 1 blue rhombus = $1/6$
 - ▲ 2 blue rhombuses = $2/6 = 1/3$
 - ▲ 1 green triangle = $1/12$
- Use the **Fraction Friction 2** handout/transparency (H4/T7) to demonstrate the process for each type to get the group started. Many will struggle until they fully understand the basic concepts of multiplication and division.
 - (#1)
 - ▲ $1/2 \cdot 1/6$ (#1)
 - Means $1/2$ of $1/6$
 - Start with the piece that represents $1/6$ (1 rhombus)
 - The question asks for “half of $1/6$ ”
 - Cover the rhombus with pieces that could be divided into two groups (triangles)
 - Take half of the pieces and record the equivalent amount (1 triangle = $1/12$)
 - So $1/2 \cdot 1/6 = 1/12$
 - ▲ $1/3 \div 1/6$ (#4)
 - Start with the piece(s) that represents $1/3$ (2 blue rhombuses)
 - State the problem in words: “How many *groups* of $1/6$ are there in $1/3$?”
 - Find the piece that represents $1/6$ (1 rhombus) and decide how many rhombuses fit on $1/3$ (2 rhombuses fit)
 - So $1/3 \div 1/6 = 2$
 - Have participants complete the handout **Fraction Friction 2** as pairs. Do not allow them to return to traditional algorithms. This exercise is intended to push participants’ understanding of these concepts in a spatial model.

**Answers:**

▲ Multiplication:

1. $1/2 \cdot 1/6 = 1/12$
One half of one sixth
2. $5/6 \cdot 1/2 = 5/12$
Five-sixths of one half
3. $1/2 \cdot 1/3 = 1/6$
One half of one third

▲ Division:

4. $1/3 \div 1/6 = 2$
How many groups of $1/6$ are there in $1/3$?
5. $1/2 \div 1/4 = 2$
How many groups of $1/4$ are there in $1/2$?
6. $3/4 \div 1/4 = 3$
How many groups of $1/4$ are there in $3/4$?

**3.3 Lecture: Common Algorithms for Multiplication and Division of Fractions**

Many of the algorithms are already apparent from analyzing patterns in the activities and discussions.

Multiplication of Fractions

Give participants time to look back over all the multiplication problems to see if they can define a rule for multiplication of fractions. Ask them to record their ideas in the Reflection section in their journals. Share ideas and check against the **Algorithms for Multiplication and Division of Fractions** transparency (T16).

Multiplication of Fractions

To multiply fractions, find the product of the numerators and the denominators.

Expanding this simple algorithm, students can reduce (simplify) the answer prior to completing the problem.

For example, $2/6 \cdot 3/12$. Use the transparency to show the following examples. This problem may be solved by multiplying 2 times 3 and then 6 times 12 to create the product of $6/72$. This product is not in simplest form because there is a common factor other than one. Using any method developed earlier, the fraction may be reduced to $6/72 = 1/12$. When problems involve large numbers or more than two fractions, it is useful to cancel within the problem before multiplying. To *cancel* (often called *cross-cancel*), any number in the numerator may be reduced with any number in the denominator.



The most common method is to look for numbers across from one another (hence the term *cross-cancel*) to be canceled. The numbers 2 and 12 are across from one another. They share a common factor of 2 (both may be reduced or divided by 2)

$$\frac{1}{6} \cdot \frac{3}{12} =$$

The process may be repeated from the numbers 3 and 6 that share a common factor of 3.

$$\frac{1}{2} \cdot \frac{1}{4} =$$

Our final answer here when multiplied is $1/12$.

Students often forget that any numerator value may be canceled with any denominator value. An alternative method is shown below. This time the 2 and 6 are being reduced, and then the 3 and 12. Note that the end result is the same.

$$\frac{1}{3} \cdot \frac{3}{12} =$$

The rules for multiplying mixed or whole numbers are the same as those for proper fractions. Mixed numbers must be changed to improper fractions before any multiplying or canceling can occur.

$$1\frac{1}{2} \cdot \frac{2}{4} = \frac{3}{1} \cdot \frac{1}{2} = \frac{3}{2}$$

Whole numbers may be treated in the same manner by making the whole into a fraction. This is easily accomplished by placing any whole number over 1.

$$5 \cdot \frac{2}{3} = \frac{5}{1} \cdot \frac{2}{3} = \frac{10}{3} = 3\frac{1}{3}$$

Point out common misconceptions about multiplying fractions. Encourage participants to share other misconceptions from their classroom experiences.

- *Common denominators*: no common denominators are needed in multiplication.
- *Canceling*: students perceive this to mean that a number goes away completely rather than to the factor of 1.
- *Cross multiplying* vs. *cross-canceling*: students often confuse canceling with cross-multiplying (which is used for proportions). Products are created from multiplying same pieces (numerators with numerators, denominators with denominators). Eliminating the word *cross-cancel* and using just *cancel* can eliminate this problem and force students to understand that any numerator can cancel with any denominator value.

Division of Fractions

The pattern for division of fractions is not as easily identified. Ask participants to look back through the division problems to see if they can identify a rule for dividing fractions.



Dividing Fractions

To divide fractions, multiply the first fraction (dividend) by the reciprocal of the second fraction (divisor).

Reciprocal

If the product of two fractions is 1, the fractions are reciprocals.

Make sure participants understand the reciprocal concept. Ask for the reciprocals of the following:

- Reciprocal of $3/4$? ($4/3$)
- Reciprocal of 2 ? ($1/2$) come from the original fraction of $2/1$
- Reciprocal of $1-3/4$? ($4/7$) comes from the original fraction of $7/4$

From the algorithm described above, a reciprocal and product are needed. This confuses students because the division problem becomes multiplication. Analyze a prior problem from the **Fraction Friction 2** handout (H4),

$$1/3 \div 1/6 =$$

A reciprocal must be found for the second fraction *only*. Use the transparency to show the work.

$$1/3 \cdot 6/1 =$$

Using canceling rules from earlier, the final product is 2. So $1/3 \div 1/6 = 2$, which confirms the concrete experience with the pattern blocks. Test this algorithm on the other problems to make sure no further questions arise.

Canceling to reduce in the problem may also be used here but *only* after the division problem has been turned into a multiplication problem. Cancellation rules only apply to multiplication.

Mixed fractions, as with multiplication, must be turned into improper fractions before the division algorithm may be applied.

Point out common misconceptions about dividing fractions. Encourage participants to share other misconceptions from their classroom experiences.

- *Common denominators*: no common denominators are required for division.
- *Canceling*: canceling can only occur after the reciprocal and product sentences have been created.
- *Flipping*: only the second fraction is “flipped;” not both fractions.
- *Division vs. multiplication*: students forget to change the problem to multiplication when they rewrite using the reciprocal because they are confused by how the division and multiplication problems are related.
- *Mixed numbers*: students divide the whole numbers and then the fractions when using mixed numbers, which is incorrect.

The most important way to understand these algorithms is the relationship between multiplication and division processes. This may also be linked to the relationship between addition and subtraction.



A useful exercise for students who continue to struggle with the algorithms is to create a chart to compare and contrast the computation processes. Students need to group related processes together as this cuts down on the number of skills to master.

4.1 Assignment #2: Fraction Bar Practice



***Note to Instructor:** You will need to decide how much time to give the class to complete their assignments so that you have time to grade, record grades and turn in materials from this course in a timely manner. If paraeducators are taking the course for credit, there will be a time limit based upon the grading period at the attending institution. You will also have to decide how you would like attendees to turn in their assignments to you, including mailing them, dropping them off at your office and so on – whatever arrangements seem to work for you and your class. **You are strongly encouraged to be firm about a completion date and may need to make some effort to follow up on attendees and their progress. Refer to the Grading Rubric handout (GR) for details on grading.**

Distribute handout **Assignment #2: Fraction Bar Practice (H12)**. Read the instructions and answer questions about how to complete the assignment. Provide the class with a date for completion and explain your process for handing in the assignment. To assist you in grading the assignment, answers to the questions are provided below.

Assignment #2: Fraction Strip Practice

This assignment is worth 150 points. The focus of this assignment is to use fraction strips to further understand adding and subtracting fractions. Not all fraction strips are present. It is important to assess if the participants can figure out the size of the missing pieces and compare them.

There are five parts to this assignment.

Step 1:

Use the attached fraction strips sheet. Cut out the entire strip for each of the 10 whole strips. Do not cut each whole apart.

Step 2: (50 points)

Fraction strips may be used to compare fractions to help student visualize the size of the fractions.

Compare $\frac{1}{3}$ and $\frac{1}{5}$. Take the thirds strip and fold all pieces back except one third. Take the fifths strip and fold back all pieces except one fifth. Compare the size of the strips.

$$\frac{1}{3} > \frac{1}{5}$$

This helps students work through the issue of the size of the number in the denominator.

Compare ($>$, $<$, $=$) the following using the above method. Explain how you found your answer or trace your pieces as proof. If no piece is available, estimate the length using other pieces.



Answers:

1. $1/4$ ____ $<$ ____ $2/4$
2. $4/5$ ____ $>$ ____ $1/6$
3. $1/11$ ____ $<$ ____ $1/10$
4. $3/6$ ____ $=$ ____ $1/2$
5. $2/3$ ____ $>$ ____ $4/7$

Step 3: (25 points)

Describe two patterns that you learned from the exercise in comparing fractions that would help you do this exercise without fraction strips.

Answers will vary, including:

- Look only at the numerator when comparing numbers with the same denominator
- As the denominator gets larger, the pieces get smaller
- Looking at the comparison of numerator to denominator can help in estimating which is larger
- Equal fractions are multiples of each

Step 4: (25 points)

During Module C, you explored how to make equivalent fractions. Find the equivalent fractions for $2/3$ using your fraction strips. Describe your process.

Answer: Line up the fraction strips and look for strips that are the same length.

$$2/3 = 4/6 = 6/9 = 8/12$$

Step 5: (50 points)

Use the following fraction addition problem. Show each step of how you would use the fraction strips to match the traditional algorithm learned in Module C.

Traditional algorithm steps:

1. Find the LCM.
 2. Create equivalent fractions for each fraction.
 3. Finish the addition problem using the common denominator.
- $$1/2 + 2/5$$

Answers:

1. Start with the $1/2$ and $2/5$ strip. Find a piece that aligns with both. That is the tenths piece. The LCM is 10.



2. Line $\frac{1}{2}$ up with the tenths. It equals $\frac{5}{10}$. Line $\frac{2}{5}$ up with tenths. It equals $\frac{4}{10}$.

$$\frac{1}{2} = \frac{5}{10}$$

$$\frac{2}{5} = \frac{4}{10}$$

3. Line up the strips to get a total of $\frac{9}{10}$.

$$\frac{1}{2} = \frac{5}{10}$$

$$+ \quad \frac{2}{5} = \frac{4}{10}$$

$$\frac{9}{10}$$



Module C

Handouts



Make Your Best Guess

For each problem below, circle the most appropriate answer. Describe how you determined your answer and include any strategies you used in detail. Do not solve the original problem.

1. $25 + 287 + 1756$ 100 1000 2000

2. $5.0076 - 3.978$ 9 2 1

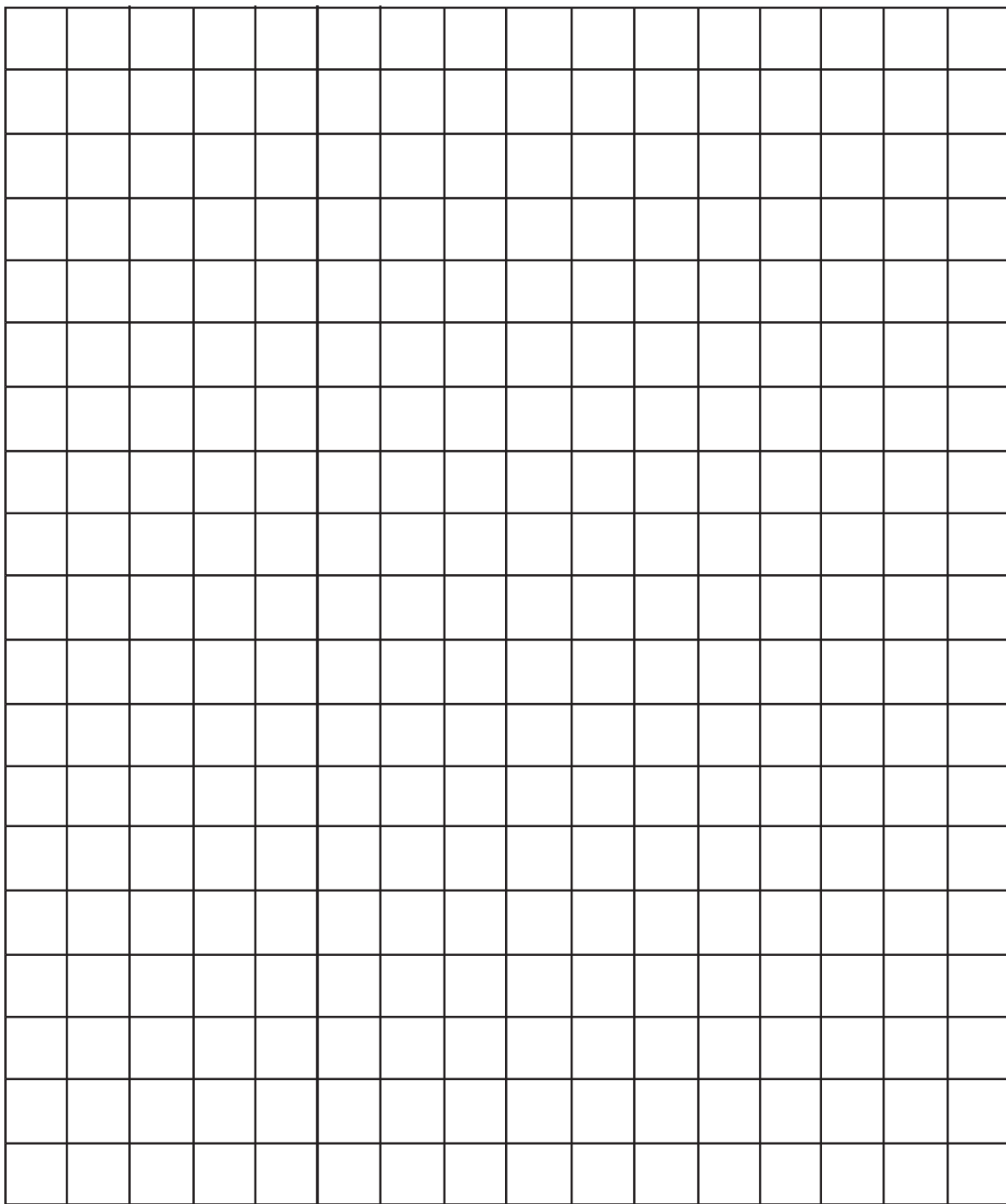
3. 1.95×2.01 0.5 2 4

4. $1 - 1/6 + 2 - 7/8 + 3/11$ 2 4 6

5. $2 - 1/5 \times 3 - 8/9$ 1 6 8



Centimeter Grid Paper





Fraction Friction

For each problem, use your pattern blocks to build each fraction and list the shapes. Complete the operation and record your progress as shown below. Make sure final answers are in the simplest form.

1. $1/2 + 3/6 =$

___ trapezoids + ___ triangles = ___ triangles + ___ triangles = ___ triangles = _____
(fraction)
= ___ hexagons = _____
(simplest form)

Or

___ trapezoids + ___ triangles = ___ trapezoids + ___ trapezoids = ___ trapezoids = _____
(fraction)
= ___ hexagons = _____
(simplest form)

2. $1/2 + 1/3 =$

_____ + _____ = _____ + _____
= _____ = _____
(fraction)

3. $1/3 + 1/6 =$

_____ = _____

4. $1/2 - 1/3 =$

_____ = _____

5. $1 - 1/3 =$

_____ = _____

Or

_____ = _____



Fraction Friction 2

For each fraction problem, complete the sentence. Then use your pattern blocks to demonstrate your work. Record your drawings to support your work. Make sure the final answers are in the simplest form.

1. $\frac{1}{2} \cdot \frac{1}{6}$ means _____ of _____

$$\frac{1}{2} \cdot \frac{1}{6} =$$

2. $\frac{5}{6} \cdot \frac{1}{2}$ means _____ of _____

$$\frac{5}{6} \cdot \frac{1}{2} =$$

3. $\frac{1}{2} \cdot \frac{1}{3}$ means _____ of _____

$$\frac{1}{2} \cdot \frac{1}{3} =$$

4. $\frac{1}{3} \div \frac{1}{6}$ means how many of _____ are in _____

$$\frac{1}{3} \div \frac{1}{6} =$$

5. $\frac{1}{2} \div \frac{1}{4}$ means _____ of _____

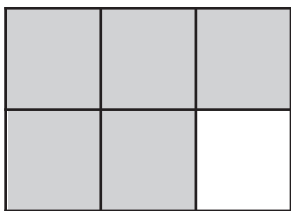
$$\frac{1}{2} \div \frac{1}{4} =$$

6. $\frac{3}{4} \div \frac{1}{4}$ means _____ of _____

$$\frac{3}{4} \div \frac{1}{4} =$$

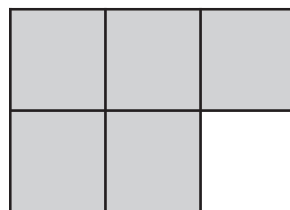
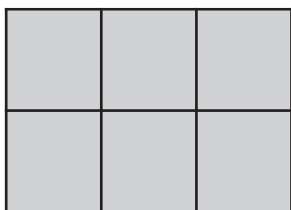
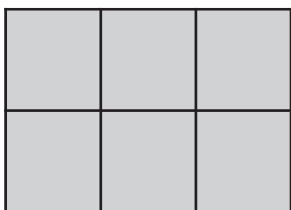


Types of Fractions



Proper Fraction:

A fraction whose numerator is smaller than the denominator.



Improper Fraction:

A fraction whose numerator is larger than or equal to the denominator.

Mixed Fraction:

A fraction that has both a whole and fractional part.

Mixed Fractions \rightarrow Improper Fractions

- Multiply the denominator by the whole number and add the numerator.
- Place over the same denominator.

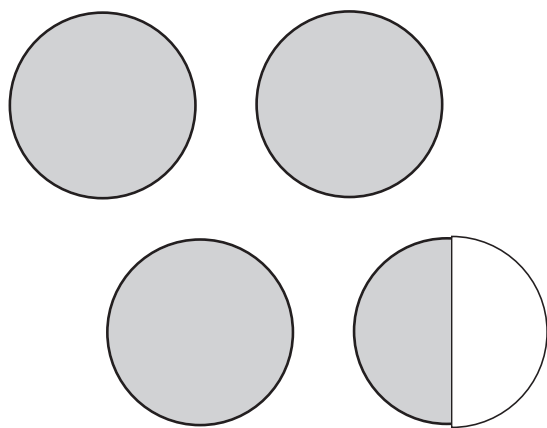
Improper Fractions \rightarrow Mixed Fractions

- Divide the denominator into the numerator.
- The quotient (answer) is the whole number.
- The remainder is the new numerator for the fractional part over the original denominator.



All Mixed Up

The wind blew through and mixed up all the fraction families. Draw lines to connect pictures to their equivalent forms.

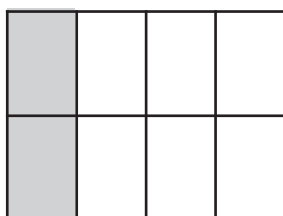
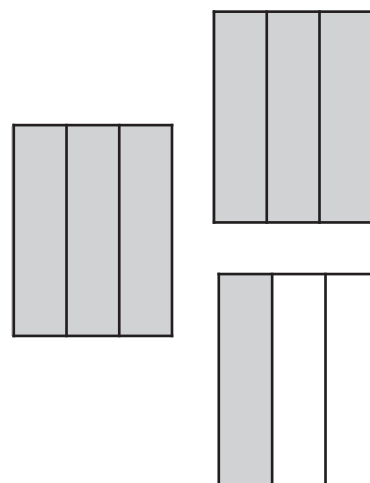


$$\frac{1}{4}$$

$$\frac{7}{5}$$

$$1\frac{2}{5}$$

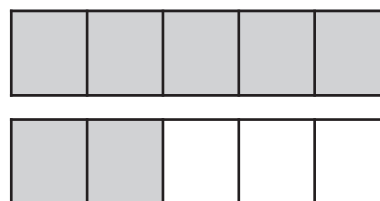
$$\frac{7}{3}$$



$$\frac{2}{8}$$

$$3\frac{1}{2}$$

$$2\frac{1}{3}$$





Fraction Bars

whole																								
halves	$\frac{1}{2}$						$\frac{2}{2}$																	
thirds	$\frac{1}{3}$				$\frac{2}{3}$				$\frac{3}{3}$															
fourths	$\frac{1}{4}$			$\frac{2}{4}$			$\frac{3}{4}$			$\frac{4}{4}$														
fifths	$\frac{1}{5}$		$\frac{2}{5}$		$\frac{3}{5}$		$\frac{4}{5}$		$\frac{5}{5}$															
sixths	$\frac{1}{6}$		$\frac{2}{6}$		$\frac{3}{6}$		$\frac{4}{6}$		$\frac{5}{6}$		$\frac{6}{6}$													
eighths	$\frac{1}{8}$		$\frac{2}{8}$		$\frac{3}{8}$		$\frac{4}{8}$		$\frac{5}{8}$		$\frac{6}{8}$		$\frac{7}{8}$		$\frac{8}{8}$									
ninths	$\frac{1}{9}$		$\frac{2}{9}$		$\frac{3}{9}$		$\frac{4}{9}$		$\frac{5}{9}$		$\frac{6}{9}$		$\frac{7}{9}$		$\frac{8}{9}$		$\frac{9}{9}$							
tenths	$\frac{1}{10}$		$\frac{2}{10}$		$\frac{3}{10}$		$\frac{4}{10}$		$\frac{5}{10}$		$\frac{6}{10}$		$\frac{7}{10}$		$\frac{8}{10}$		$\frac{9}{10}$		$\frac{10}{10}$					
twelfths	$\frac{1}{12}$		$\frac{2}{12}$		$\frac{3}{12}$		$\frac{4}{12}$		$\frac{5}{12}$		$\frac{6}{12}$		$\frac{7}{12}$		$\frac{8}{12}$		$\frac{9}{12}$		$\frac{10}{12}$		$\frac{11}{12}$		$\frac{12}{12}$	



Adding Fractions with Unlike Denominators

$$\frac{1}{3} + \frac{1}{4}$$

$$\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

1. Find the LCM for the denominators.

Use any of the methods discussed in Number Theory.

The LCM is 12.

2. Set up equivalent fractions for the new denominators.

$$\begin{array}{r} \frac{1}{3} = \frac{\quad}{12} \\ + \frac{1}{4} = \frac{\quad}{12} \\ \hline \end{array}$$

$$\frac{1 \cdot 4}{3 \cdot 4} = \frac{\quad}{12} \qquad \frac{1 \cdot 3}{4 \cdot 3} = \frac{\quad}{12}$$

3. Solve the new addition problem.

$$\begin{array}{r} \frac{1}{3} = \frac{4}{12} \\ + \frac{1}{4} = \frac{3}{12} \\ \hline \frac{7}{12} \end{array}$$

$$\frac{4}{12} + \frac{3}{12} = \frac{7}{12}$$



Subtracting Fractions with Unlike Denominators

$$2/3 - 1/6$$

1. Find the LCM for the denominators.

2. Set up equivalent fractions for the new denominators.

3. Solve the new subtraction problem.



Simplifying Fractions

Simplest Form:

The numerator and denominator have no common factor except 1.

$\frac{2}{8}$ Not in the simplest form

$\frac{2 \div 2}{8 \div 2} = \frac{1}{4}$ Simplest form

$\frac{12}{36}$ Not in the simplest form



Multiplication and Division Concepts

1. Write the related mathematical addition sentences.
 $3 \cdot 4 =$
2. Write the verbal interpretation of the multiplication problem $3 \cdot 4 =$
3. Write the related addition sentence in words and as a mathematical sentence.
 $5 \cdot 2/3 =$
4. Solve the above addition problem to get the answer for $5 \cdot 2/3$.
5. Write the related addition sentence in words and as a mathematical sentence.
 $2/3 \cdot 6$
6. Write the related verbal interpretation of this problem.
 $2/3 \cdot 1/4$
7. Solve using the patterns observed from the demonstration.
 $2/3 \cdot 1/4$
8. Try using the problem $1/2 \cdot 1/3$. Show with the paper-folding method and confirm with the multiplication rules from above.



9. Write the verbal interpretation of the division problem $12 \div 3$.

10. Write the verbal interpretation of the division problem $1 \div 1/2$ and the related repeated subtraction problem.

11. Write the verbal interpretation of the division problem $2/3 \div 3$. Draw the paper-folding diagram to show this process.



Name _____

Assignment #2: Fraction Bar Practice

The following assignment is worth 150 points.

The focus of this assignment is to use fraction bars to further understand adding and subtracting fractions.

There are five parts to this assignment.

Step 1:

Use the attached fraction bars sheet. Cut out the entire bar for each of the 10 whole bars. Do not cut each whole apart.

Step 2: (50 points)

Fraction bars may be used to compare fractions to help students visualize the size of the fractions.

Compare $\frac{1}{3}$ and $\frac{1}{5}$. Take the thirds bar and fold all pieces back except one third. Take the fifths bar and fold back all pieces except one fifth. Compare the size of the bars.

$$\frac{1}{3} > \frac{1}{5}$$

This helps students work through the issue of the size of the number in the denominator.

Compare ($>$, $<$, $=$) the following using the above method. Explain how you found your answer or trace your pieces as proof. If no piece is available, estimate the length using other pieces.

1. $\frac{1}{4}$ _____ $\frac{2}{4}$

2. $\frac{4}{5}$ _____ $\frac{1}{6}$

3. $\frac{1}{11}$ _____ $\frac{1}{10}$

4. $\frac{3}{6}$ _____ $\frac{1}{2}$

5. $\frac{2}{3}$ _____ $\frac{4}{7}$



Step 3: (25 points)

Describe two patterns you learned from the exercise in comparing fractions that would help you do this exercise without fraction bars.

Step 4: (25 points)

During Module C, you explored how to make equivalent fractions. Find the equivalent fractions for $\frac{2}{3}$ using your fraction bars. Describe the process you used.

Step 5: (50 points)

Use the following fraction addition problem. Show each step of how you would use the fraction bars to match the traditional algorithm you learned in Module C.

Traditional algorithm steps:

1. Find the LCM.
2. Create equivalent fractions for each fraction.
3. Finish the addition problem using the common denominator.

$$\frac{1}{2} + \frac{2}{5}$$



Fraction Bars

whole												
halves	$\frac{1}{2}$						$\frac{2}{2}$					
thirds	$\frac{1}{3}$				$\frac{2}{3}$				$\frac{3}{3}$			
fourths	$\frac{1}{4}$			$\frac{2}{4}$			$\frac{3}{4}$			$\frac{4}{4}$		
fifths	$\frac{1}{5}$		$\frac{2}{5}$		$\frac{3}{5}$		$\frac{4}{5}$		$\frac{5}{5}$			
sixths	$\frac{1}{6}$		$\frac{2}{6}$		$\frac{3}{6}$		$\frac{4}{6}$		$\frac{5}{6}$		$\frac{6}{6}$	
eighths	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	$\frac{8}{8}$				
ninths	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{6}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{9}{9}$			
tenths	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$	$\frac{10}{10}$		
twelfths	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{4}{12}$	$\frac{5}{12}$	$\frac{6}{12}$	$\frac{7}{12}$	$\frac{8}{12}$	$\frac{9}{12}$	$\frac{10}{12}$	$\frac{11}{12}$	$\frac{12}{12}$



Module C

Transparencies



Module Goals

Module C: Number Sense and Fractional Concepts

The paraeducator will:

- **Apply number theory concepts to represent numbers in a variety of ways**
- **Use number sense to justify the reasonableness of solutions for a variety of computation and problem-solving situations**
- **Use concrete materials to develop fractional concepts for addition and subtraction**
- **Use concrete materials to develop fractional concepts of multiplication and division**

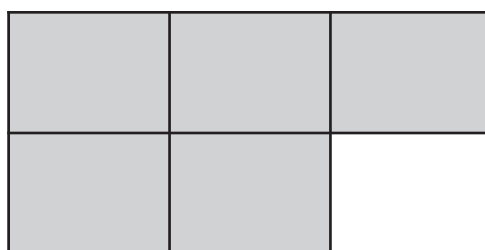
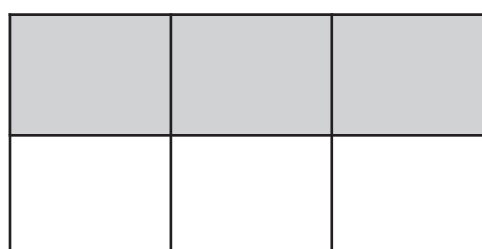
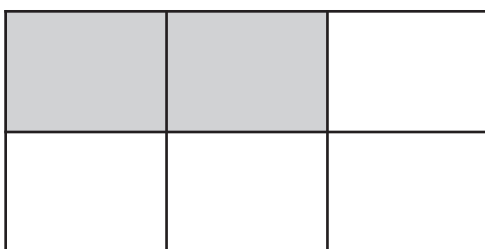


Number Sense

The ability to evaluate whether an answer is reasonable and appropriate.



Adding Fractions



Rules for Adding Fractions with Like Denominators:

-

-

-



Subtracting Fractions with Unlike Denominators

$$\frac{2}{3} - \frac{1}{6}$$

1. Find the LCM for the denominators.
2. Set up equivalent fractions for the new denominators.
3. Solve the new subtraction problem.



Fraction Friction

For each problem, use your pattern blocks to build each fraction and list the shapes. Complete the operation and record your progress as shown below. Make sure the final answers are in the simplest form.

1. $\frac{1}{2} + \frac{3}{6} =$

___ trapezoids + ___ triangles = ___ triangles + ___ triangles = ___ triangles = _____
(fraction)

= ___ hexagons = _____
(simplest form)

Or

___ trapezoids + ___ triangles = ___ trapezoids + ___ trapezoids = ___ trapezoids = _____
(fraction)

= ___ hexagons = _____
(simplest form)

2. $\frac{1}{2} + \frac{1}{3} =$

_____ + _____ = _____ + _____

= _____ = _____
(fraction)

3. $\frac{1}{3} + \frac{1}{6} =$

_____ = _____

4. $\frac{1}{2} - \frac{1}{3} =$

_____ = _____

5. $1 - \frac{1}{3} =$

_____ = _____

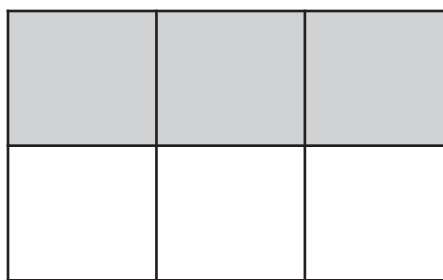
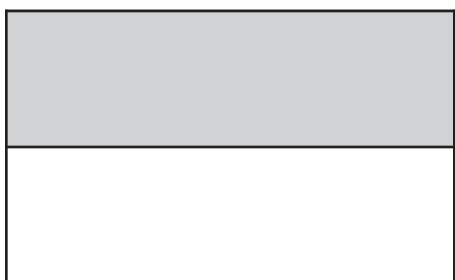
Or

_____ = _____



Unlike Denominators

$$\frac{1}{2} + \frac{3}{6} =$$



$$\frac{3}{6} + \frac{3}{6} = \frac{6}{6} = 1$$



Fraction Friction 2

For each fraction problem, complete the sentence. Then use your pattern blocks to demonstrate your work. Record your drawings to support your work. Make sure the final answers are in the simplest form.

1. $\frac{1}{2} \cdot \frac{1}{6}$ means _____ of _____

$$\frac{1}{2} \cdot \frac{1}{6} =$$

2. $\frac{5}{6} \cdot \frac{1}{2}$ means _____ of _____

$$\frac{5}{6} \cdot \frac{1}{2} =$$

3. $\frac{1}{2} \cdot \frac{1}{3}$ means _____ of _____

$$\frac{1}{2} \cdot \frac{1}{3} =$$

4. $\frac{1}{3} \div \frac{1}{6}$ means _____ of _____

$$\frac{1}{3} \div \frac{1}{6} =$$

5. $\frac{1}{2} \div \frac{1}{4}$ means _____ of _____

$$\frac{1}{2} \div \frac{1}{4} =$$

6. $\frac{3}{4} \div \frac{1}{4}$ means _____ of _____

$$\frac{3}{4} \div \frac{1}{4} =$$



Fraction Friction Reflection

•

•

•

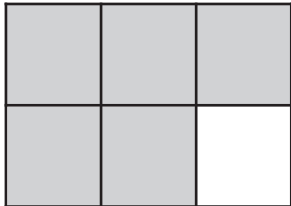
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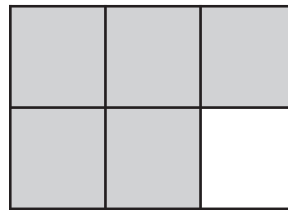
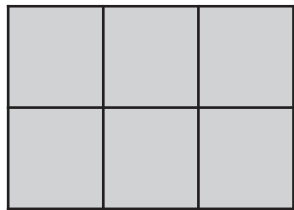
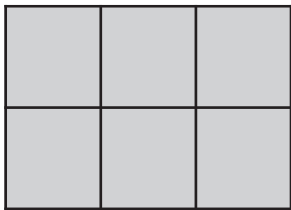


Types of Fractions



Proper Fraction:

A fraction whose numerator is smaller than the denominator.



Improper Fraction:

A fraction whose numerator is larger than or equal to the denominator.

Mixed Fraction:

A fraction that has both a whole and a fractional part.



Types of Fractions

Mixed Fractions \rightarrow Improper Fractions

- Multiply the denominator by the whole number and add the numerator.
- Place over the same denominator.

Improper Fractions \rightarrow Mixed Fractions

- Divide the denominator into the numerator.
- The quotient (answer) is the whole number.
- The remainder is the new numerator for the fractional part over the original denominator.



Fraction Bars

whole												
halves	$\frac{1}{2}$						$\frac{2}{2}$					
thirds	$\frac{1}{3}$				$\frac{2}{3}$				$\frac{3}{3}$			
fourths	$\frac{1}{4}$			$\frac{2}{4}$			$\frac{3}{4}$			$\frac{4}{4}$		
fifths	$\frac{1}{5}$		$\frac{2}{5}$		$\frac{3}{5}$		$\frac{4}{5}$		$\frac{5}{5}$			
sixths	$\frac{1}{6}$		$\frac{2}{6}$		$\frac{3}{6}$		$\frac{4}{6}$		$\frac{5}{6}$		$\frac{6}{6}$	
eighths	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	$\frac{8}{8}$				
ninths	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{6}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{9}{9}$			
tenths	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$	$\frac{10}{10}$		
twelfths	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{4}{12}$	$\frac{5}{12}$	$\frac{6}{12}$	$\frac{7}{12}$	$\frac{8}{12}$	$\frac{9}{12}$	$\frac{10}{12}$	$\frac{11}{12}$	$\frac{12}{12}$



Adding Fractions with Unlike Denominators

$$\frac{1}{3} + \frac{1}{4}$$

$$\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

1. Find the LCM for the denominators.

Use any of the methods discussed in number theory.

The LCM is 12.



Adding Fractions with Unlike Denominators

2. Set up equivalent fractions for the new denominators.

$$\begin{array}{r} 1/3 = \quad /12 \\ + \quad 1/4 = \quad /12 \\ \hline \end{array}$$

$$\frac{1 \cdot 4}{3 \cdot 4} = \frac{\quad}{12} \qquad \frac{1 \cdot 3}{4 \cdot 3} = \frac{\quad}{12}$$

3. Solve the new addition problem.

$$\begin{array}{r} 1/3 = 4/12 \\ + \quad 1/4 = 3/12 \\ \hline \quad \quad \quad 1 \\ \quad \quad \quad 7/12 \end{array}$$

$$4/12 + 3/12 = 7/12$$



Equivalent Fractions

Two fractions that cover the same portion of a whole but are written with a different, but equal, fraction.

$$1/2 =$$

Other examples: $1/3 = 2/6 = 3/9$



Subtracting Fractions with Like Denominators

Rules for Subtracting Fractions with Like Denominators:

-
-
-



Simplifying Fractions

Simplest Form:

The numerator and denominator have no common factor except 1.

$\frac{2}{8}$ Not in the simplest form

$\frac{2 \div 2}{8 \div 2} = \frac{1}{4}$ Simplest form

$\frac{12}{36}$ Not in the simplest form



Multiplication and Division Concepts

$$3 \cdot 4 = \quad (\text{numbers 1 \& 2})$$

$$5 \cdot 2/3 = \quad (\text{numbers 3 \& 4})$$

$$2/3 \cdot 6 = \quad (\text{number 5})$$

$$2/3 \cdot 1/4 = \quad (\text{number 6})$$

$$12 \div 3 = \quad (\text{number 9})$$

$$1 \div 1/2 = \quad (\text{number 10})$$

$$2/3 \div 3 = \quad (\text{number 11})$$



Algorithms for Multiplication and Division of Fractions

Multiplication of Fractions

To multiply fractions, find the product of the numerators and the denominators.

Dividing Fractions:

To divide fractions, multiply the first fraction (dividend) by the reciprocal of the second fraction (divisor)

Reciprocal:

If the product of two fractions is 1, the fractions are reciprocals.



Module D

Instructor's Guide



Module D: Decimal and Percent Concepts

A. Module Goals

Using the **Module Goals** transparency (T1), review the goals of the Module D.

Module D: Decimal and Percent Concepts

The paraeducator will:

- Use concrete materials to develop decimal concepts
- Use concrete materials to develop percentage concepts
- Connect decimal and percentage concepts



Goal 1: Use concrete materials to develop decimal concepts.



1.1 Discussion: Review Addition and Subtraction of Decimals

The paraeducator will use prior experience to identify the rules for addition and subtraction of decimals.

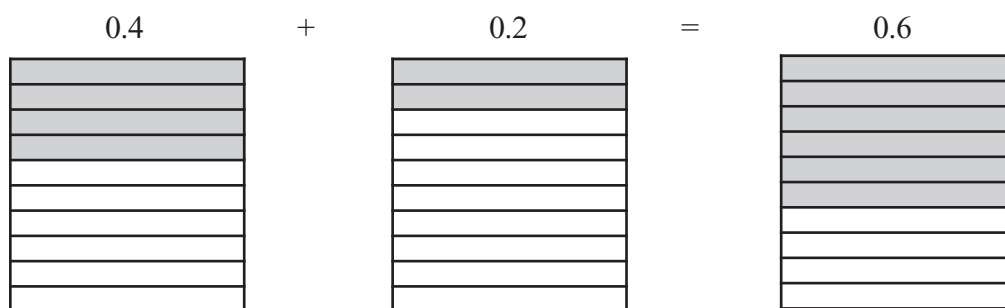
Materials:

- **Adding and Subtracting Decimals** transparency and handout (T2/H1)
- **Decimal Paper** handout and transparency (H2/T3)



1.1.1 Steps

- Ask participants to complete the following problem in their journals: $\$1.25 + \$3.56 = (\$4.81)$.
- Most participants, like students, do well adding and subtracting decimals because of experience with money from the early elementary grades.
- How to add decimals may be shown with the same shaded rectangle model as used for fractions. This model solidifies why we align decimals when we add or subtract. Have participants write the number sentence to match the model on **Adding and Subtracting Decimals** handout (H1).



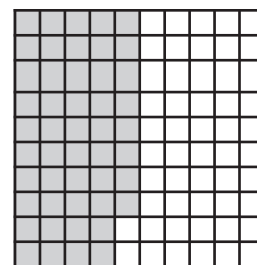
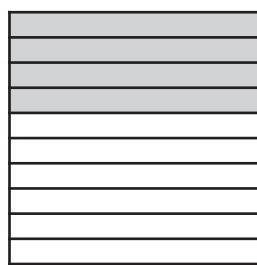
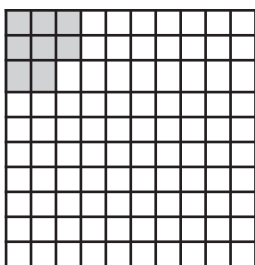


- Decimal grids represent the place value (tenths) and then combine together using the common grid until the grid is filled up. This is identical to adding with common denominators. Add the information below to the transparency.

$$4/10 + 2/10 = 6/10 \quad \text{or} \quad \begin{array}{r} 0.4 \\ + 0.2 \\ \hline 0.6 \end{array}$$

- Adding with uncommon place value is similar to adding with unlike denominators. Have participants write the number sentence to match the model.

$$0.08 \quad + \quad 0.4 \quad = \quad 0.48$$



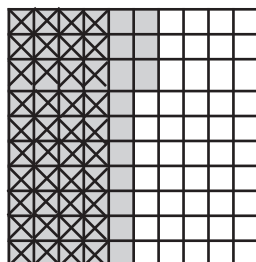
- In fraction form, this may be written as

$$8/100 + 40/100 \quad \text{or} \quad \begin{array}{r} 0.08 \\ + 0.4 \\ \hline 0.48 \end{array}$$

- Using prior skills developed for adding fractions, the common denominator would be 100. Show work on transparency.

$$8/100 + 40/100 = 48/100 \quad \text{or} \quad \begin{array}{r} 0.08 \\ + 0.4 \\ \hline 0.48 \end{array}$$

- This results in the same answer as the model. The example shows how important aligning the decimal becomes when adding decimals. Aligning decimals accommodates for the place-value differences.
- The same is true for subtracting decimals. Have participants shade the model to represent the decimal sentence: $0.53 - 0.4$. Use the transparency/handout **Decimal Paper (T3/H2)**.
 - ▲ Shade 0.53 (remind participants that with subtraction – we start with total)
 - ▲ Cross out 0.4 (it is important to note that tenths are entire columns or are equal to 40 hundredths)





- ▲ This leaves 0.13 or 13 hundredths as demonstrated in the model

$$\begin{array}{r} 0.53 \\ - 0.40 \\ \hline 0.13 \end{array}$$

- ▲ Note that it helps to align the decimals by adding *trailing zeros*. This also makes the denominator change in terms of fractions to a common denominator (it creates an equivalent amount).



1.2 Activity: Grids Galore

The paraeducator will use grids to demonstrate multiplication and division of decimals.

Materials:

- Handout and transparency **Grids Galore (H3/T4)**



1.2.1 Steps

- Ask participants to work in pairs.
- Have them complete the **Grids Galore** handout; remind participants of the shading method used for fractions. Using **the Grids Galore** transparency, go over the examples for multiplication and division to make sure everyone understands.
- Discuss the results as a group.
 1. $0.4 \times 0.7 = 0.28$
 2. $0.7 \times 0.6 = 0.42$
 3. $0.1 \times 0.7 = 0.07$
 4. $0.8 \div 4 = 0.2$
0.8 divided into four equal groups. Each group contains 0.2 (2 tenths).
 5. $0.28 \div 0.07 = 4$
0.28 divided into groups of 0.07. There are four equal groups.
 6. $0.36 \div 9 = 0.04$
0.36 divided into nine equal groups. Each group contains 0.04 (4 hundredths).
- Numbers 1-3 should remind participants of fraction multiplication.
- Numbers 4-6 require participants to look at the divisor (number dividing) before deciding how to divide up the given decimal amount. This is a challenging step in understanding decimal division.



1.3 Discussion: Creating Decimal Rules

The paraeducator will use the prior activity to generalize rules for multiplication and division of decimals.

Materials:

- **Multiplication of Decimals** transparency and handout (T5/H4)



1.3.1 Steps

Multiplication of Decimals

- To be successful with multiplication, students must understand how to rename place values. Refer to the transparency **Multiplication of Decimals**.
- Start with a simple decimal multiplication problem: 4×1.7 .
- This may be solved by returning to repeated addition. Add this problem to the transparency/handout **Multiplication of Decimals**: $1.7 + 1.7 + 1.7 + 1.7 = 6.8$.
 - ▲ This is an effective method for finding the product of a whole number and a decimal.
 - ▲ This is not effective for finding the product of two decimals.
- As a group, put the following digits in the proper place-value column. Record on the transparency/handout **Multiplication of Decimals**.
 - ▲ 8 hundredths
 - ▲ 29 hundredths
 - ▲ 29 tenths
 - ▲ 134 hundredths

Tens	Ones	.	Tenths	Hundredths
		.	0	8
		.	2	9
	2	.	9	
	1	.	3	4

- The last two problems will cause paraeducators the most confusion, as it does students.
- If participants struggle, write it as a fraction to determine if the fraction is proper (less than one) or improper (greater than one).
- Answers as fractions:
 - ▲ $8/100$ (less than one)
 - ▲ $29/100$ (less than one)
 - ▲ $29/10$ (greater than one $2-9/10$)
 - ▲ $134/100$ (greater than one $1-34/100$)
- The original problem 4×1.7 may be translated in terms of place value. Show this on the transparency.

$$\begin{array}{r} 17 \text{ tenths} \\ \times 4 \\ \hline 68 \text{ tenths} \end{array}$$

 and $68 \text{ tenths} = 6.8$
 - ▲ Note that the final answer had one decimal place after the decimal point.



- Show a second example of 3.45×5 on the transparency **Multiplication of Decimals**. Repeated addition may be used as 5 is a whole number. Show the place-value version to help with decimal placement

$$\begin{array}{r} 345 \text{ hundredths} \\ \times 5 \\ \hline 1725 \text{ hundredths} \end{array} \quad \text{and} \quad 1725 \text{ hundredths} = 17.25$$
- ▲ Note that the final answer had two decimal places after the decimal point.
- Moving to multiplication with two decimals is often best accomplished using the fraction equivalents. Write the decimal problem 0.3×0.4 in terms of the equivalent fraction. (Use the **Multiplication of Decimals** transparency.)

$$3/10 \cdot 4/10 = 12/100 = 0.12$$
- This means that $0.3 \times 0.4 = 0.12$.
- Present one last problem: 0.4×0.46 . Repeat with the fraction equivalents. (Use the **Multiplication of Decimals** transparency.)

$$4/10 \cdot 46/100 = 184/1000 = 0.184$$
- This means that $0.4 \times 0.46 = 0.184$.
- Return to the **Grids Galore (H3)** handout to see if any rules may be generalized.
- Ask participants to generalize about the rules for multiplying decimals. Record any rules generated on the transparency/handout **Multiplication of Decimals (T5/H4)**.
 - ▲ Multiply numbers as for normal multiplication.
 - ▲ Count the number of decimals to the right of the decimal point in each factor.
 - ▲ The final answer should have the same total amount of decimal places.
- Point out to participants that students can explore rules with calculators by putting in several decimal problems and recording the outcome. They can generalize rules from these experiences without being overwhelmed by the “hand” multiplication.

Division of Decimals

- From the **Grids Galore** handout, look for patterns in division. Use the transparency/handout **Division of Decimals (T6/H5)**.
- Problems 4 and 6 on the handout **Grids Galore (H3)** appear as normal division, holding the decimal point in place. This is because of the whole number.

$$\begin{array}{r} 0.2 \\ 4 \overline{) 0.8} \end{array}$$
- The decimal point should be placed; then normal division procedures should be followed.
- The difficulty comes in explaining division with two decimal numbers.

$$0.28 \div 0.07 \text{ is } 0.07 \overline{) 0.28}$$
 - ▲ From the **Grids Galore** handout, the answer was 4.
 - ▲ It appears that regular division took place but the decimals have disappeared.



- If this problem is written with fractions, $0.28 \div 0.07$ is $28/100 \div 7/100$. Add the following steps to the transparency.
- Using the fraction division from earlier activities, the problem may be rewritten as multiplication and reduced.

$$\frac{28}{100} \cdot \frac{100}{7} = \frac{28}{7} = 4$$

- ▲ While this has proven the answer correct, it has not given an algorithm for decimal division with two decimals.
- From the fractions above, participants should recognize that both 28 and 7 are treated as whole numbers once the fraction parts have canceled. If this could be applied to decimal division, the process could be simplified.
- If the divisor (0.07) could be made into a whole number, the division could easily be performed with skills already developed from early-elementary grades.

$$.07 \overline{) 0.28}$$

- Multiplying by 100 would make 0.07 a whole number. We know that to alter one portion of a fraction, the same operation must be performed on the other portion to maintain equivalent fractions.

$$\frac{0.28}{0.07} \cdot \frac{100}{100} = \frac{28}{7} = 4$$

- The same result occurs when the decimal is moved two places to the right in both the *divisor* (0.07) and the *dividend* (0.28). The divisor determines how many places the decimal must move in both portions.

$$.07 \overline{) 0.28} = 7 \overline{) 28}$$

- When the decimal point still exists under the division sign, it is placed in the quotient (answer) first; then regular division algorithms can be used. Use the following problem as an example. (Use the transparency **Division of Decimals (T6)**; letter c:

$$1.3 \overline{) 2.197}$$

- ▲ Look at the divisor (1.3) to decide how many places to move the decimal to make it a whole number (1).
- ▲ Move the decimal in the dividend (2.197) the same number of places.
- ▲ Place the remaining decimal in the answer (quotient).

$$13 \overline{) 21.97}$$

- ▲ Divide using the regular division algorithm.

$$\begin{array}{r} 1.69 \\ 13 \overline{) 21.97} \\ \underline{-13} \\ 89 \\ \underline{-78} \\ 117 \\ \underline{-117} \\ 0 \end{array}$$



- ▲ Check the plausibility of the answer using number reuse skills as follows. To check division, multiply the two factors. The answer is a little more than 1.5 multiplied by 13. This can estimated to be $13 + 6.5 \approx 19.5$. This sum puts the calculation close to 21.97, so we assume our answer is on the right track.
- The final decimal division concept is when the answer is *non-terminating* (does not end). Students need to have a sense of how far to go in the calculation.
- Altering the above problem slightly, from working the problem out to a few decimal places, it appears that the answer will not end (using transparency **Division of Decimals**; letter d):
 - ▲ The teacher generally decides to what place value the answer is rounded. The problem should continue one place value past the required place value.
 - ▲ To make that happen, zeros must be added to the dividend (21.96).
 - ▲ The reason for adding zeros is to provide a “place” for the problem to occur.
 - ▲ Remember: Trailing zeros do not change the actual value of the decimal.
 - ▲ Keep applying the division algorithm until the answer can be rounded. For the following problem, round to the nearest hundredth.

$$\begin{array}{r}
 1.689 \\
 13 \overline{) 21.960} \\
 \underline{- 13} \\
 89 \\
 \underline{- 78} \\
 116 \\
 \underline{- 104} \\
 120 \\
 \underline{- 117} \\
 30
 \end{array}$$

- ▲ The problem can stop midway once the required place value is determined.
- ▲ The final answer is ≈ 1.69 (*Note: the approximate symbol is used when rounding occurs*).



Goal 2: Use concrete materials to develop percentage concepts.

The paraeducator will use mental math to see the common knowledge used in percentage calculations.



2.1 Discussion: Percentage Concepts

2.1.1 Steps

- Students conceptually understand the value of percentages without much instruction because it aligns with their knowledge of normal whole numbers.
 - ▲ Fifty percent must be less than ninety percent.



- ▲ A seventy-percent chance of rain is very likely whereas a thirty-percent chance is not.
- ▲ Some current elementary mathematics texts teach percentage concepts before decimals and fractions. The fraction and decimal lessons are based primarily on percentages. The philosophy behind this approach is to use the common knowledge of percentages to build more difficult concepts.
- To emphasize the point above, ask the participants to do the following exercise mentally using the transparency **Thinking Percentages (T7)**.
 - ▲ Compute the following mentally (Burns, 1992):
 - 100% of \$200 (200)
 - 50% of \$200 (100)
 - 25% of \$200 (50)
 - 10% of \$200 (20)
 - 5% of \$200 (10)
 - 1% of \$200 (2)
- The most difficult aspect of percentage is to help students understand the meaning of 100%. While it seems obvious that 100% means a perfect score or the total, it is more difficult when asked to find 100% of something.
- The remainder of the percentage problems above were based on the relationships between the percentages.
 - ▲ Fifty percent relied on knowledge that 50% is one half.
 - ▲ Twenty-five percent was half of that amount.
 - ▲ The difficult leap is 10%. Ask participants what strategies they used to figure out 10%.
 - They knew that 10 tens made a hundred, so they divided the \$200 by 10 to get 20.
 - They actually did 1% first, knowing there were one hundred ones, giving \$2 as 1%. Ten percent came from multiplying the result by 10, giving \$20.
- Students need several opportunities to conceptually understand percentages before moving on to mathematical definitions. An example follows in the next activity.



2.2 Activity: Area of What?

The paraeducator will use estimation skills to determine the shaded percent of irregular shapes.

Materials:

- **Percentage Patterns** handout (**H6**) (1 set per group of 4)
- **Percentage Overlay** transparency (**T8**) (1 per pair)



2.2.1 Steps

- Divide the class into groups of four.
- Provide each group with a set of **Percentage Patterns** handout.



- Ask each group to estimate the percent shaded for each pattern.
- Have participants record their estimations in their journals.



2.3 Lecture: Defining Percentages

Use the **Defining Percentages** transparency (T9).

The percentage symbol (%) means “out of one hundred.”

If the shaded areas could be measured against a “hundreds” measure, the estimations could be checked.

Ask participants what they might use.



2.4 Activity: Continued

- Provide each pair (or participant) with a copy of the **Percentage Overlay** transparency.
- Have them overlay the transparency on their shape and “count” the squares in the shaded area. This is the percentage shaded as there are 100 squares on the grid.
- Questions will arise about how to count squares that are not completely covered.
- Lead a discussion about how to “create” a square from several partial squares. While this is estimation, it will still provide a fairly accurate percent of the shaded area.
- Compare answers as a group; answers will vary.
- Starting with an area of irregular shapes is a fun way to attract students’ attention, but it also visually emphasizes the percentage concept of “part of the whole (100%).”
- Activity centers may be used so that students are not confined to paper models for their concept of percentage (100 beads on a string with a certain percentage of the beads in one color, shaded rectangular grid patterns where the shading is scattered; shading of non-rectangular shapes, etc.).
- Followup activities for students may include having them draw and shade their own shapes to be a specified percentage. They can then trade with partners to check their answers using 100-grid overlays.



Goal 3: Connect decimal and percentage concepts.



3.1 Activity: Name It

The paraeducator will compare models and the equivalent decimal and percentage values.

Materials:

- **Math Mystery** handout (H7)



3.1.1 Steps

- Have participants work in pairs or individually.
- Have paraeducators complete the handout **Math Mystery**
 - ▲ $0.21 = 21\%$



- ▲ $0.4 = 0.40 = 40\%$
- ▲ Shade 37 squares = 37%
- ▲ $1.0 = 1 = 100\%$
- ▲ Shade 1 square = 0.01



3.2 Discussion: Communicating Links Between Decimals and Percentages

The paraeducator will use prior experience to determine connections between percentages and decimals.



3.2.1 Steps

- From the **Math Mystery** handout, have participants look for patterns between the decimal form and the percentage model. Ask them to record their observations in the Reflection section of their journals.
 - ▲ If the decimal is hundredths, the digits tell how many squares are shaded
 - ▲ If the decimal is only tenths, a trailing zero may be added to show the number of square shaded
- Have participants look for patterns between the decimal form and the percentage form.
 - ▲ The digits in each are the same
 - ▲ The percentage model includes the same digits without the decimal
 - ▲ The decimal seems to move two places to the right each time when going from decimal to percentage
- Have participants look for patterns between the percentage form and the percentage model.
 - ▲ The digits show how many squares are shaded out of one hundred
- Looking for patterns encourages students to create their own conclusions about the links between decimals and percentages.
 - ▲ In reality, decimals and percentages represent the same data in different formats
 - ▲ Different formats are necessary to represent a variety of applications
- Students who commonly struggle with these concepts have weak links in one or more of the following:
 - ▲ “Whole” concept (as in one whole)
 - ▲ Decimal place value
 - ▲ Definition of percentage
 - ▲ Ability to assess the size of percentages
- Ask students to describe the patterns and relationships in words and pictures so the teacher can assess their communication of the concept relationships.



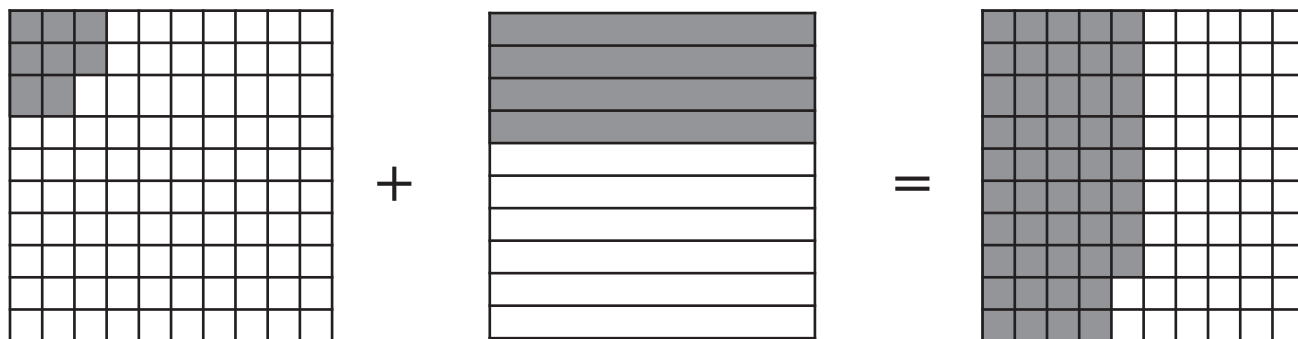
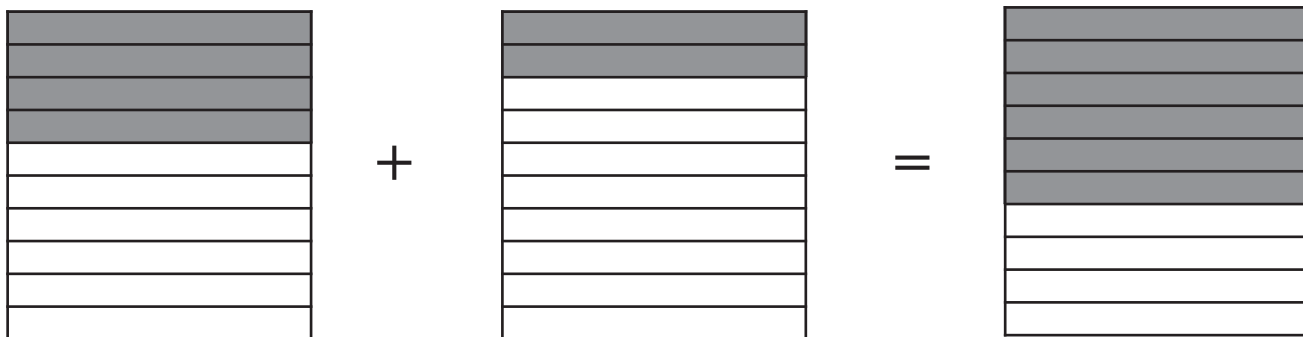
Module D

Handouts



Adding and Subtracting Decimals

Write the appropriate decimal problem for this model.



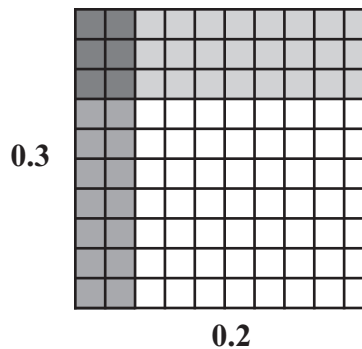


Decimal Paper



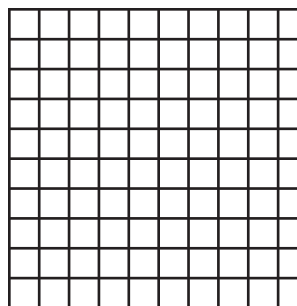
Grids Galore

Example: $0.3 \times 0.2 = 0.06$

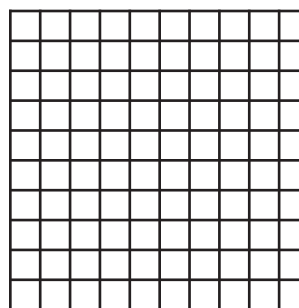


Complete each decimal multiplication by shading the decimal values and looking for overlapping squares.

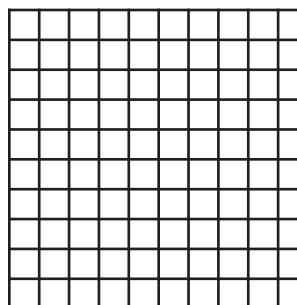
1. $0.4 \times 0.7 =$



2. $0.7 \times 0.6 =$



3. $0.1 \times 0.7 =$

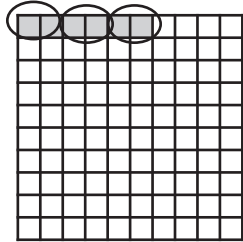




Grids Galore

Examples:

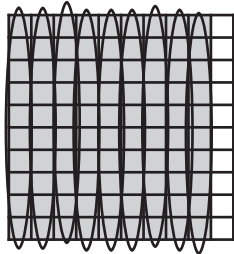
$$0.06 \div 3 =$$



$$0.06 \div 3 = 0.02$$

Says 6 hundredths divided into 3 equal groups.
How many are in each group?

$$0.90 \div 0.10 =$$



$$0.90 \div 0.10 = 9$$

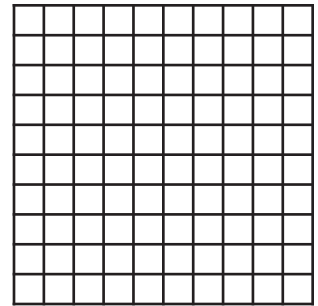
Says 90 hundredths divided into groups of 10 hundredths.
How many groups can be created?

Study the examples above. Complete the following sentences and demonstrate the solution using the grids.

4. $0.8 \div 4 =$

_____ divided into _____ equal groups.

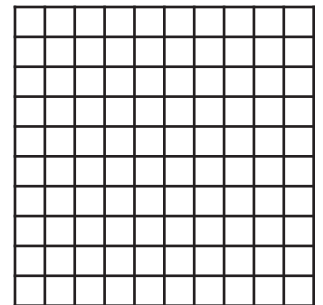
Each group contains _____ .



5. $0.28 \div 0.07 =$

_____ divided into equal groups of _____.

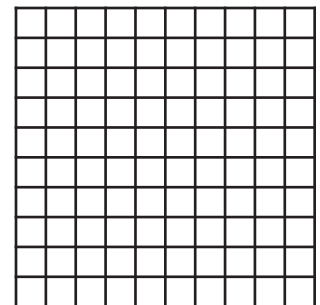
There are _____ equal groups.



6. $0.36 \div 9 =$

_____ divided into _____ equal groups.

Each group contains _____ .





Multiplication of Decimals

Write the related addition sentence. Solve the problem. 4×1.7

Place the following in the chart below.

- 8 hundredths
- 29 hundredths
- 29 tenths
- 134 hundredths

Tens	Ones	.	Tenths	Hundredths
		.		
		.		
		.		
		.		

Solve:

$$4 \times 1.7$$

$$3.45 \times 5$$

$$0.3 \times 0.4$$

$$0.4 \times 0.46$$



Division of Decimals

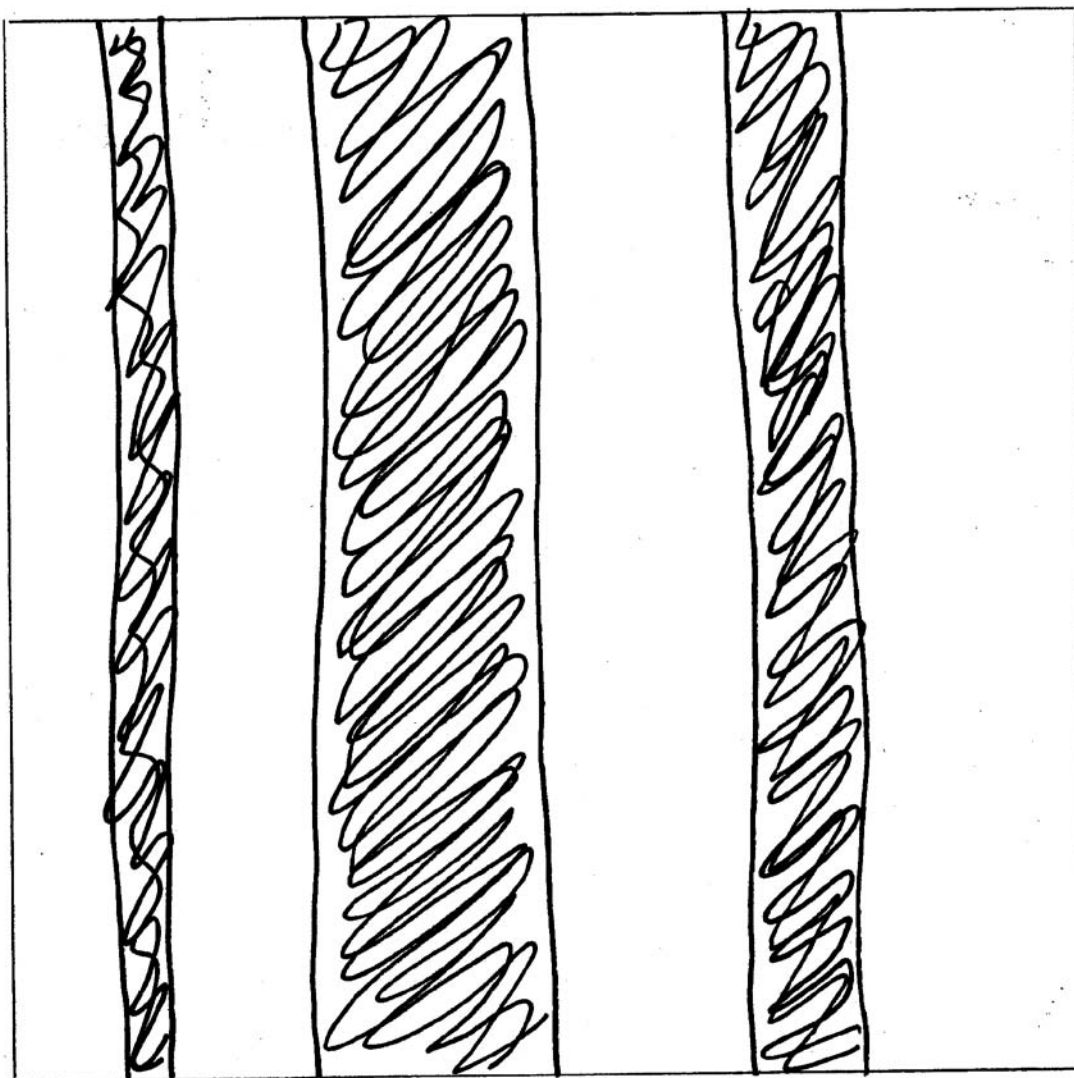
a)
$$\begin{array}{r} 0.2 \\ 4 \overline{) 0.8} \end{array}$$

**When dividing by a whole number, place the decimal.
Follow regular rules for division.**

b) $0.28 \div 0.07$ is written $0.07 \overline{) 0.28}$



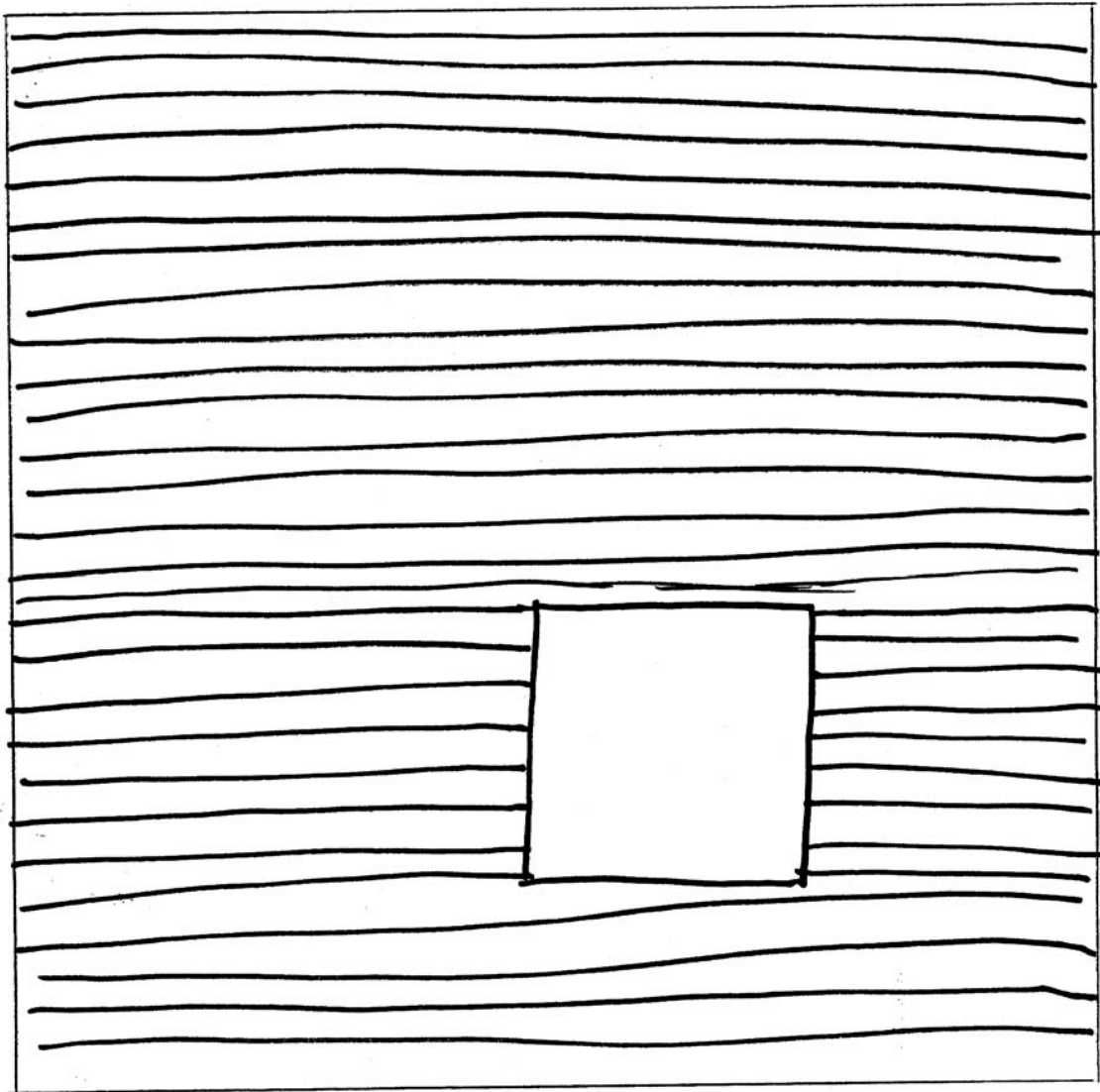
Percentage Patterns



Number 1



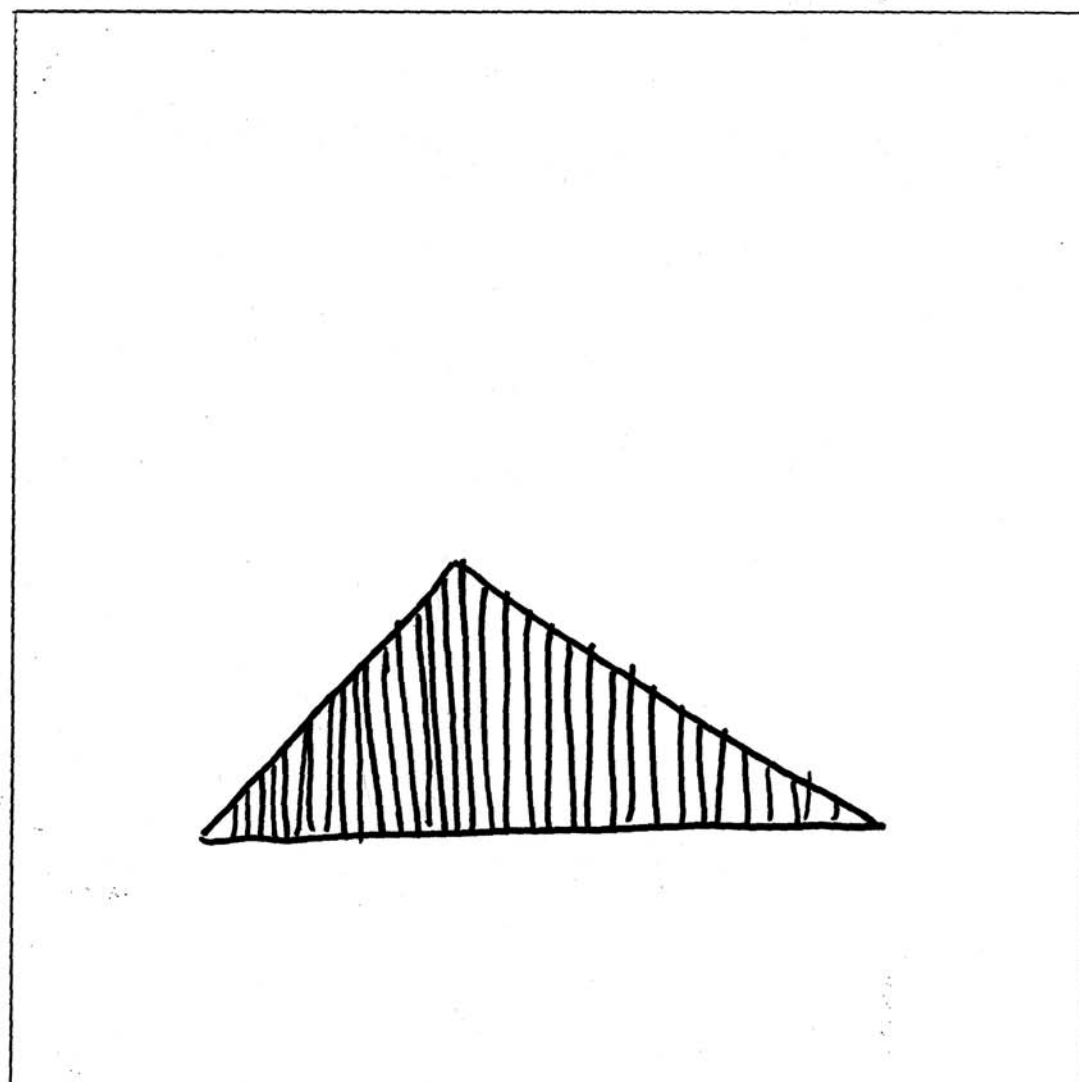
Percentage Patterns



Number 2



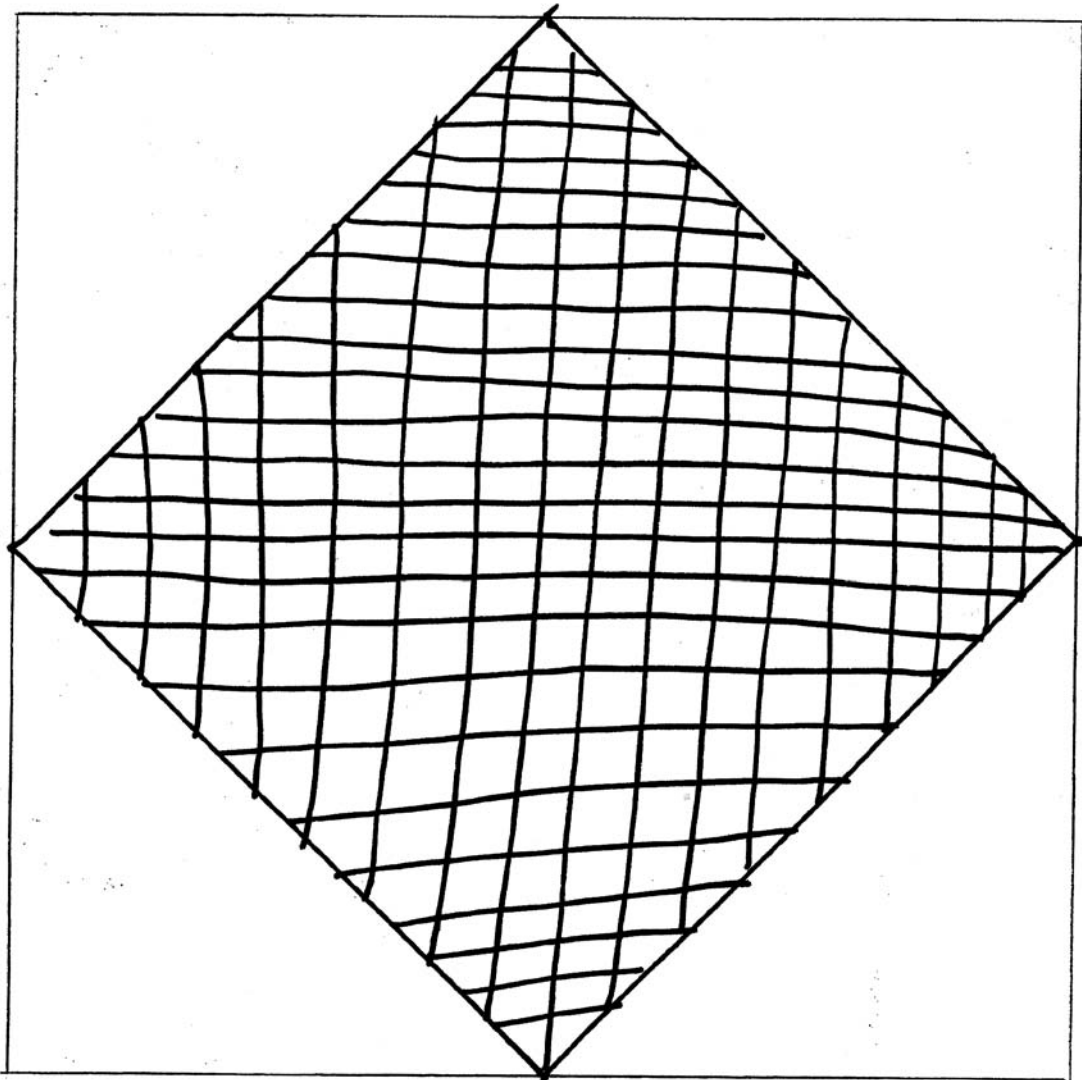
Percentage Patterns



Number 3



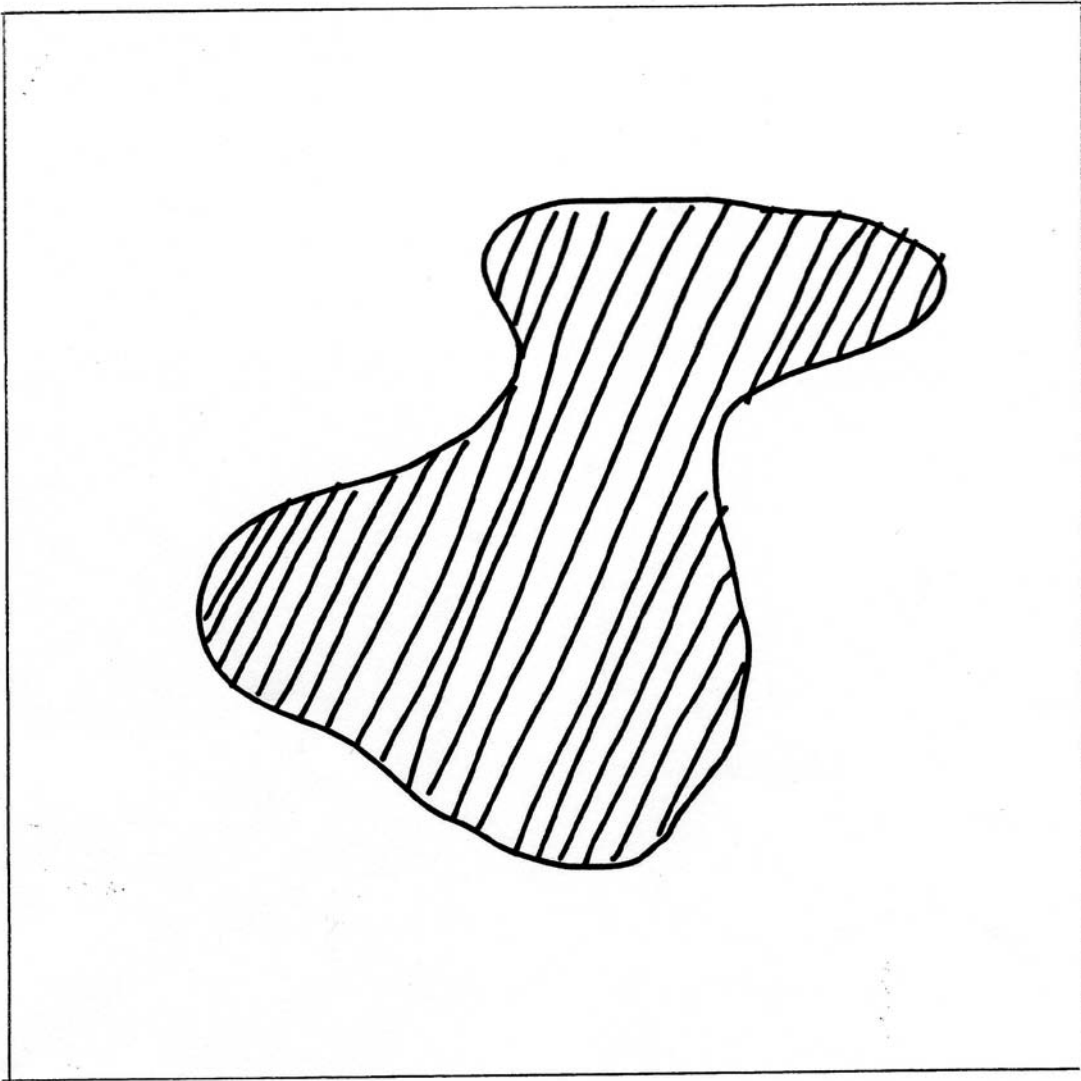
Percentage Patterns



Number 4



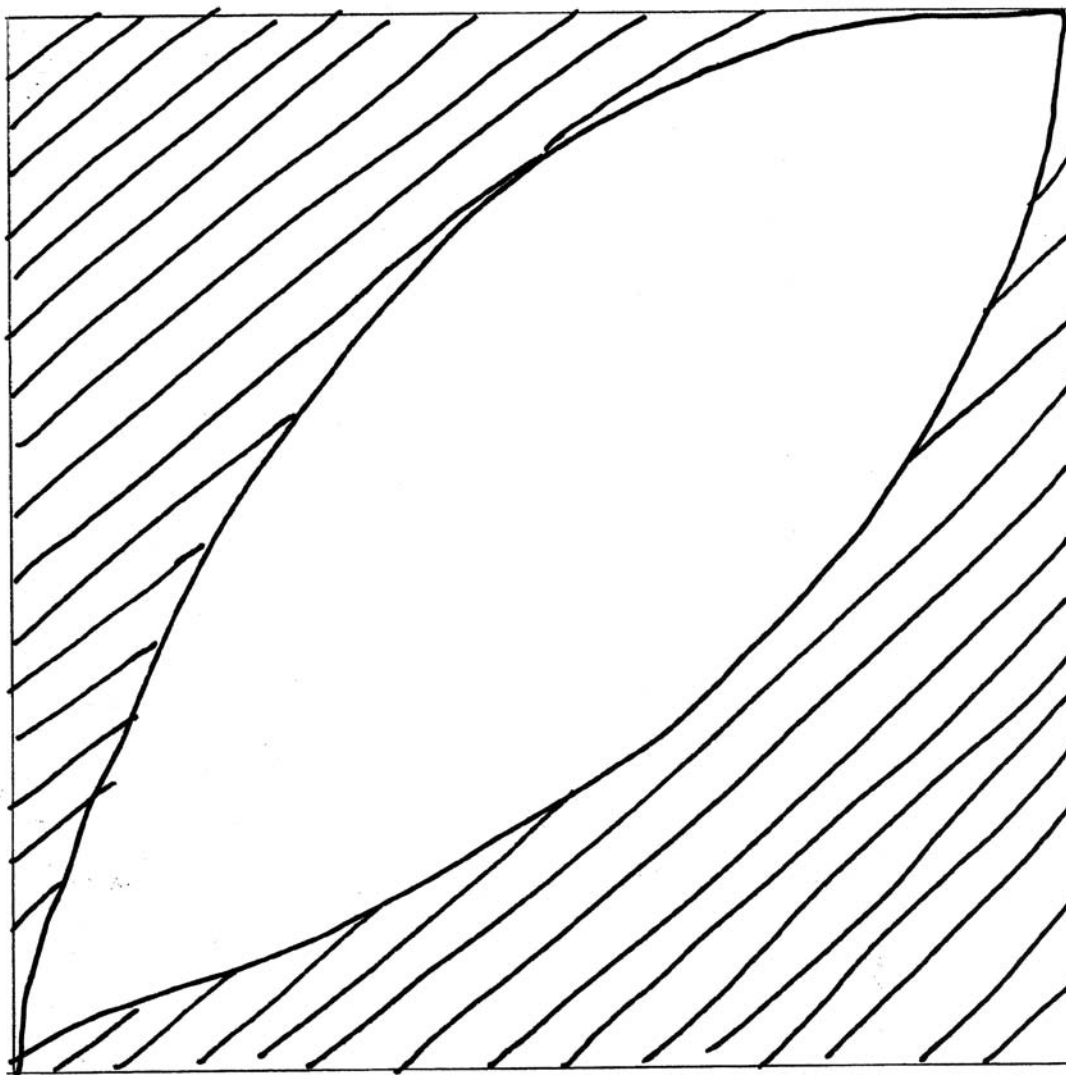
Percentage Patterns



Number 5



Percentage Patterns



Number 6

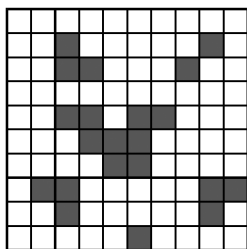


Math Mystery

Eli the Math Detective is trying to solve the mystery relating percentages and decimals. Fill in the missing clues to help Eli solve his math mystery.

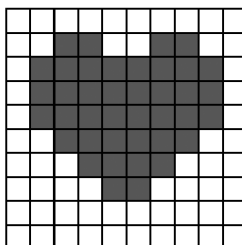
Decimal

0.21

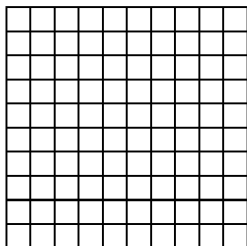


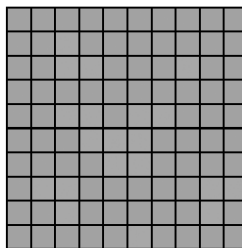
Percentage

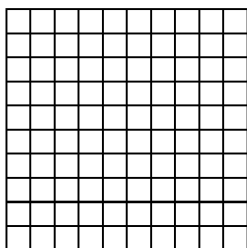
0.4



0.37







1%



Module D

Transparencies



Module Goals

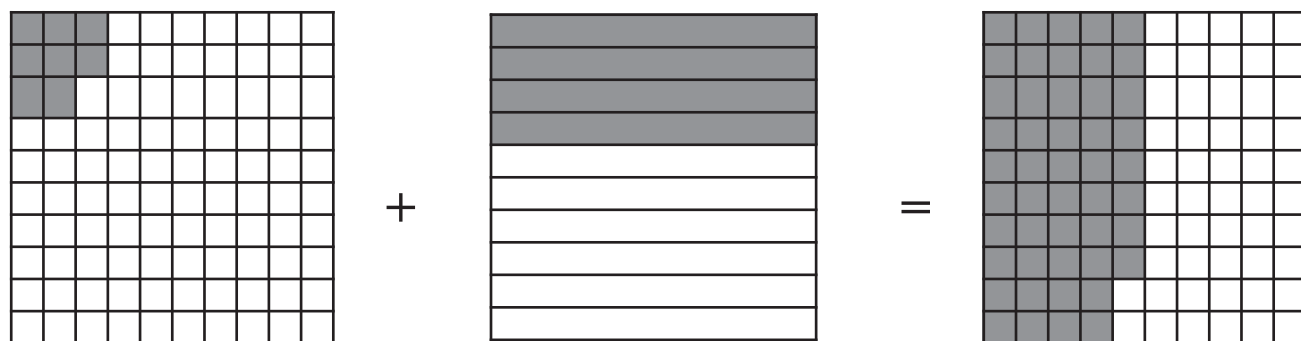
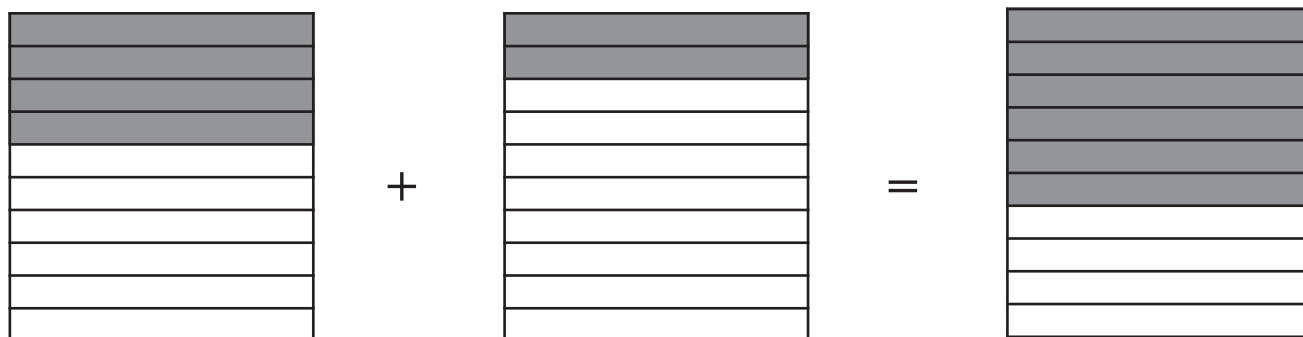
Module D: Decimal and Percentage Concepts

The paraeducator will:

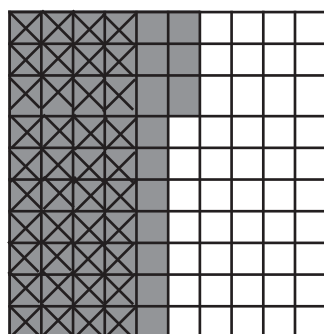
- **Use concrete materials to develop decimal concepts**
- **Use concrete materials to develop percentage concepts**
- **Connect decimal and percentage concepts**



Adding and Subtracting Decimals



$$0.53 - 0.4$$

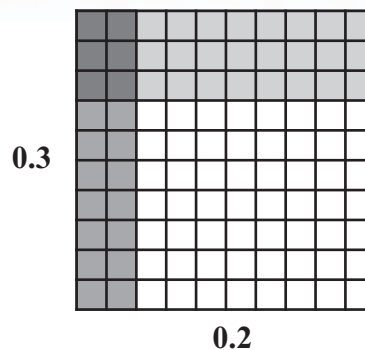


[illegible]



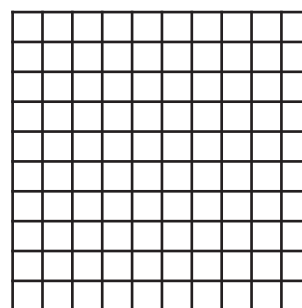
Grids Galore

Example: $0.3 \times 0.2 = 0.06$

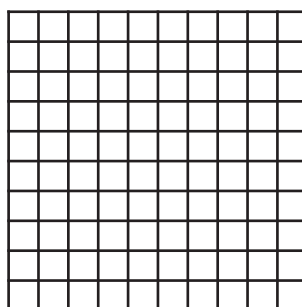


Complete each decimal multiplication by shading the decimal values and looking for overlapping squares.

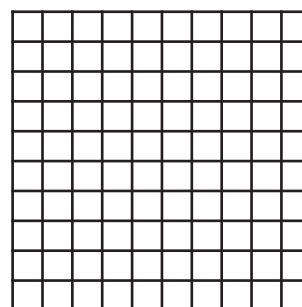
1. $0.4 \times 0.7 =$



2. $0.7 \times 0.6 =$



3. $0.1 \times 0.7 =$

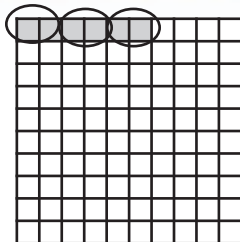




Grids Galore

Examples:

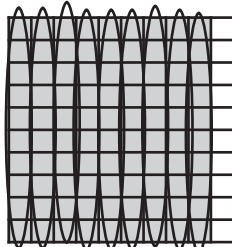
$$0.06 \div 3 =$$



$$0.06 \div 3 = 0.02$$

Says 6 hundredths divided into three equal groups
How many are in each group?

$$0.90 \div 0.10 =$$



$$0.90 \div 0.10 = 9$$

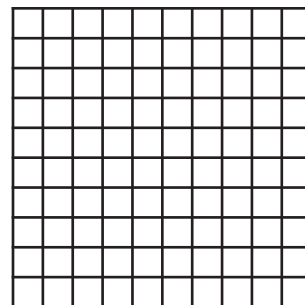
Says 90 hundredths divided into groups of 10 hundredths.
How many groups can be created?

Study the examples above. Complete the following sentences and demonstrate the solution using the grids.

4. $0.8 \div 4 =$

_____ divided into _____ equal groups.

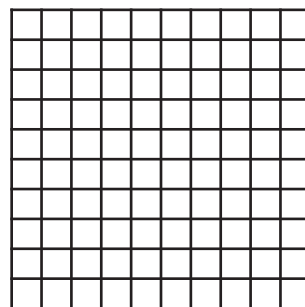
Each group contains _____ .



5. $0.28 \div 0.07 =$

_____ divided into equal groups of _____.

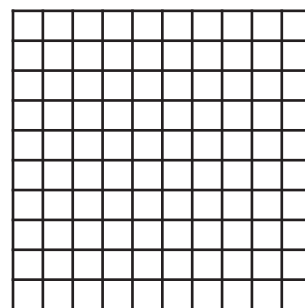
There are _____ equal groups.



6. $0.36 \div 9 =$

_____ divided into _____ equal groups.

Each group contains _____ .





Multiplication of Decimals

$$4 \times 1.7 =$$

Tens	Ones	.	Tenths	Hundredths
		.		
		.		
		.		
		.		
		.		

$$4 \times 1.7 =$$

$$3.45 \times 5 =$$



$$0.3 \times 0.4 =$$

$$0.4 \times 0.46 =$$

Rules generalization:

-
-
-



Division of Decimals

a)
$$\begin{array}{r} 0.2 \\ 4 \overline{)0.8} \end{array}$$

When dividing by a whole number, place the decimal.
Follow the regular rules for division.

b) $0.28 \div 0.07$ is written $0.07 \overline{)0.28}$

c) $1.3 \overline{)2.197}$

d) $1.3 \overline{)2.1960}$



Thinking Percentages

100% of \$200

50% of \$200

25% of \$200

10% of \$200

5% of \$200


1% of \$200



Percentage Overlay



Defining Percentage



**The percentage symbol (%) means
“out of one hundred.”**



Module E

Instructor's Guide



Module E: Making Connections

A. Module Goals

Use the **Module Goals** transparency (T1) to review the goals of Module E.

Module E: Making Connections

The paraeducator will:

1. Develop conversion algorithms among fractions, decimals, and percentages
2. Compare rational and irrational numbers through equivalent forms using a variety of strategies
3. Apply computational procedures for fractions, decimals and percentages to real-world problems



Goal 1: Develop conversion algorithms among fractions, decimals, and percentages.



1.1 Lecture: Decimals and Percentages

The previous module ended with a discovery of the connection between decimals and percentages. The basic examples included in the last activity may be completed easily with grid models. As students progress, the decimal and percentage concepts become more complex. Algorithms are useful in helping students become more efficient.

The algorithm for changing a *percentage to a decimal* should already have come up in the observations from the activity. That is, participants should have noticed that the decimal appeared to move to the left from the location of the percentage symbol. Remind participants that any whole number has a “hidden” decimal at the end. For example, 25 may be written as 25.0 without changing the value.

Return to the definition of percentage: “out of one hundred.” This is a clue that the percentage number is technically being divided by one hundred.

$$50\% = 50/100$$

If the fraction is said verbally, we would say “fifty hundredths.” This implies a decimal less than one. We expect this result because 50% is less than 100%. Fifty hundredths as a decimal is 0.50 or 0.5, which implies that 50 hundredths is equivalent to 5 tenths. From this example and those in the last activity in Module D, the decimal moves two places to the left.

$$50\% = 50/100 = 0.50 = 0.5$$



*** Note to Instructor:** Encourage participants to copy examples and details into their journals for the following conversions. These are not being supplied on a handout because the final activity for this goal requires participants to make personal sense of all the conversion algorithms. Participants should not be copying information from the transparency. They should be writing their own observations making sure the details are correct. Write necessary examples next to the transparency definitions.



Use the **Decimals-Percentages-Fractions** transparency (T2). Ask participants to verbalize their algorithms.

Percentage → Decimal

- Conceptually divide by 100.
- Move the decimal two places to the *left*.

The algorithm for changing a *decimal to a percentage* should be similar. From the activity, it appeared that the decimal always moved two places to the right to form the percentage symbol. This would mean multiplying by 100. We know that this is valid because the percentage is based on one hundred. Use the **Decimals-Percentages-Fractions** transparency.

Decimals → Percentage

- Multiply the decimal by 100.
- Move the decimal two places to the *right* and add the percentage symbol (%).

For both concepts it is important to think about the resulting answer rather than just memorizing the algorithm. Number sense again comes into play because students should have an idea whether the initial decimal is greater or less than 1 and how that affects the percentage equivalent.

If a decimal is greater than 1 initially, the percentage is over 100. Conversely, a percentage greater than 100 should have a decimal greater than 1. The same relationship applies to decimals less than 1.

Middle-school students are presented problems that have a decimal in the percentage such as 25.6%. Those who struggle with the conversion have only memorized the algorithm and not understood the concept. This will be addressed later in this goal.



1.2 Discussion: Fractions and Percentages

The paraeducator will use percentage knowledge with equivalent fractions to determine an algorithm for conversion between the two forms.



1.2.1 Steps

- In grades K-4, students have made connections between common fractions, decimals, and percentages such as $0.50 = 50\% = \frac{1}{2}$. The concept of a dollar is a strong model used in early grades. For the above example, the one hundred base is 100 pennies. So, 50% of those pennies is 50¢ or a half dollar.
- To work with percentages such as 37%, the money example may be used to show 37¢ or 0.37. The fraction representation can be confusing.
- The conversion for *percentage to fraction* is generally the easiest because the definition of percentage is “out of 100.” This is by definition a fraction.



- Ask participants to verbally describe each conversion from percentage to fraction. Make sure their final answers are in the simplest form. Use the **Decimals-Percentages-Fractions** transparency (T2).

- ▲ $25\% = ?$ ($25/100 = 1/4$)
- ▲ $37\% = ?$ ($37/100$)
- ▲ $100\% = ?$ ($100/100 = 1$)
- ▲ $3\% = ?$ ($3/100$)
- ▲ $5.5\% = ?$ ($55/1000 = 11/200$)

Note: See next point before presenting

- Most participants will not know what to do with the last scenario because it has a decimal. Students share this confusion especially when asked to convert this percentage to a decimal. Some students will argue that nothing needs to be done, but they forget that this is still a percentage.
 - ▲ Start with the same concept of putting the percentage over 100.
 $5.5\% = 5.5/100$.
 - ▲ In mathematics, a decimal within a fraction is a confusing form (called a *complex fraction*).
 - ▲ A whole number is required in both the numerator and the denominator.
 - ▲ Ask participants what they would need to multiply the numerator by to make it a whole number. (multiply by 10).
 $5.5\% = 5.5/100 \cdot 10/10 = 55/1000$
 - ▲ Participants should recognize this as a very small number that matches the small percentage with which we started. Reducing this fraction to lowest terms gives
 $5.5\% = 5.5/100 \cdot 10/10 = 55/1000 = 11/200$.
 - ▲ While this is a very strange fraction, students can reach the end result using what they know about equivalent fractions.
 - ▲ How to check the validity of this answer will be covered later in this goal.
- Use the **Decimals-Percentages-Fractions** transparency (T2) to complete the rule. Ask participants to verbalize their algorithms.

Percentage → Fraction

- ▲ Place the percentage over 100 and drop the % symbol.
- ▲ Reduce the fraction when necessary.

- Converting from *fractions to percentages* may be more difficult, depending on the denominator of the fraction.
- If the denominator is easily changed into 100, the fraction is easily converted to a percentage. The fraction $4/5$ already tells us that the percentage should be more than 50% because 4 is more than half of 5. We also know that it should be quite large because 4 out of 5 is almost the entire amount.



- We could guess at fractions with 100 in the denominator but that could take a long time and become frustrating. Instead, by using prior knowledge of equivalent fractions, we can focus our thinking.

$$\frac{4}{5} = \frac{\quad}{100}$$
 - ▲ The new denominator would be 100 because the definition of percent is “out of 100.”
 - ▲ Participants should see that the original denominator needs to be multiplied by 20; the same must occur with the numerator.

$$\frac{4}{5} \cdot \frac{20}{20} = \frac{80}{100}$$
 - ▲ From the final fraction $\frac{4}{5} = 80\%$.
 - ▲ This matches our predictions using number sense.
 - ▲ This is a common fraction stated in advertising when the ad states, “Four out of five doctors agree.” It tells us that five doctors could have been interviewed or 100 doctors could have been interviewed to arrive at the same final percentage.
- This process becomes more difficult if the denominator does not go into 100 evenly, such as the fraction $\frac{1}{8}$. This will be addressed in the next discussion.
- Use the transparency **Decimals-Percentages-Fractions** to complete the rule. Ask participants to verbalize their algorithms. This will be further expanded in the next discussion. (Keep the second dark bullet covered at this time)

Fraction → Percentage

- ▲ If the denominator can easily be converted into 100, set up an equivalent fraction with 100 as the new denominator.
- ▲ Name the percent from the numerator.



1.3 Discussion: Fractions and Decimals

The paraeducator will use knowledge of decimal division to create an algorithm for converting fractions and decimals.



1.3.1 Steps

- Of the conversions, fractions and decimals conversions are often the most difficult because a variety of skills are required.
- Converting from *decimals to fractions* is generally the easiest.
- Similar to the approach for converting percentages to decimals, the conversion relies on the definition. This time it is the definition of place value for decimals.
- Given a decimal, create a fraction from its place value. The numerator includes the digits. The denominator is the place value.

$$0.45 = 45 \text{ hundredths} = \frac{45}{100}$$
 - ▲ Make sure to reduce the fraction.

$$\frac{45}{100} = \frac{9}{20}$$
 - ▲ This fraction is a surprise because it does not resemble the original decimal or fraction.



- Decimals greater than one whole can be approached two different ways. For example, 1.35 is greater than one.
 - ▲ 1.35 may be rewritten as above using
$$1.35 = 135 \text{ hundredths} = 135/100 = 27/20.$$
 - ▲ Note that this is an improper fraction, implying a value greater than 1. This matches our expectation.
 - ▲ Changing the improper fraction to a mixed fraction would be
$$27/20 = 1-7/20.$$
- A second approach is to leave off the whole portion for the initial calculation because we know that the ending fraction will contain that whole. Remind participants that the decimal states “one and thirty-five hundredths.”
 - ▲ Using only the fractional portion
$$35/100 = 7/20.$$
 - ▲ The final answer is again $1-7/20$.
 - ▲ This method is much shorter but relies on a complete understanding of the relationships between fractions and decimals and the meaning of whole numbers for fractions and decimals.
- Use the transparency **Decimals-Percentages-Fractions** to complete the rule. Ask participants to verbalize their algorithms.

Decimal → Fraction

- ▲ Change the decimal into its related place value fraction.
- ▲ Reduce the fraction.
- ▲ For fractions greater than 1:
 - Convert the decimal to an improper fraction, OR
 - Use only the fractional portion and add on the whole for the final answer.
- Converting *fractions to decimals* is more difficult. If the denominator of the fraction can easily divide one hundred, equivalent fractions may be made, which is simple to convert into decimal or percentage form.
- The difficulty arises when the denominator does not easily divide one hundred. Use the following fraction: $1/8$
 - ▲ Since 8 does not divide one hundred, equivalent fractions cannot be used.
 - ▲ When fractions were first introduced in early-elementary grades, the idea of division should have been introduced as well. Fraction notation is another form of division notation. The division problem $2\overline{)10}$ may be written as the fraction $10/2$. The solution to either form is 5. As the fraction was improper, a solution greater than 1 was expected.
 - ▲ With a proper fraction such as $1/8$, the equivalent decimal should be less than 1.
 - ▲ To create equivalent decimals, we must divide the *denominator into the numerator*.
$$\begin{array}{r} 8 \overline{)1} \end{array}$$



- ▲ A decimal must be added in order to complete the division. Have participants complete the division by hand.

$$\begin{array}{r} 0.125 \\ 8 \overline{)1.000} \\ \underline{-8} \\ 20 \\ \underline{-16} \\ 40 \\ \underline{-40} \\ 0 \end{array}$$

- ▲ As this decimal *terminates* (ends), the decimal equivalent for one eighth is 0.125. This decimal is less than 1, which meets our expected result.
- ▲ Other useful terminating decimals to commit to memory include $1/2 = 0.5$, $1/4 = 0.25$, and $3/4 = 0.75$. Students usually have learned these from early-elementary instruction or prior money experiences.
- While the example above included a decimal that ended, many decimals go on infinitely. One type is a *repeating decimal*. Have participants complete the following problem by hand to see the pattern.

$$1/3 = 3 \overline{)1.0}$$

$$\begin{array}{r} 0.333 \\ 3 \overline{)1.000} \\ \underline{-9} \\ 10 \\ \underline{-9} \\ 10 \\ \underline{-9} \\ 1 \end{array}$$

- ▲ For this problem, it is obvious that this pattern will continue. We call this decimal a *repeating decimal* (goes on infinitely in a repeating pattern). The special notation for repeating decimals is listed below. The two fractions listed below should be committed to memory as they are commonly used.
- $$1/3 = 0.\overline{3} \qquad 2/3 = 0.\overline{6}$$
- ▲ Some decimals, best found with a calculator, have large repeating patterns.
- $$7/13 = 0.\overline{538461} = 0.538461538461538461.....$$
- ▲ The *repeat bar* is placed over the part of the pattern that is repeating.
Note: The decimal should never be rounded unless specified by a teacher or problem.
 - A second special decimal is a *non-repeating, non-terminating decimal* called an *irrational number*. This implies that the decimal does not end but also does not repeat a pattern.
 - ▲ The fractional approximation for π is an example:
- $$\pi = 22/7 = 3.142857143....$$



- ▲ This pattern goes on infinitely but does not repeat. This decimal would need to be rounded with instruction from a teacher.
Note: Any time rounding occurs, always use the approximate symbol (\approx).
- Use the **Decimals-Percentages-Fractions** transparency to complete the algorithm. Ask participants to verbalize their algorithms.

Fraction \rightarrow Decimal

- ▲ Divide the denominator into the numerator.
 - We can also return to the *fraction to percentage conversion* and add the following.

Fraction \rightarrow Percentage

- ▲ If the denominator can easily be converted to 100, set up an equivalent fraction.
 - Name the percentage from the numerator.
- ▲ If the denominator cannot be easily be converted to 100, change the number to a decimal, and then convert the decimal to a percentage.



1.4 Activity: Making It Stick

The paraeducator will practice with the new algorithms to convert fractions, decimals, and percentages.

Materials:

- Large bag of M&M's or Smarties
- Handout **Making It Stick (H1)**
- Handout **Making It Stick-ier (H2)**
- Colored pencils, markers, or crayons



1.4.1 Steps

- Provide each participant with the handout **Making It Stick (H1)**. Students, like the participants, often receive large numbers of rules and definitions. Until the algorithms for these conversions are internalized, the student will not be successful at fully understanding and working with these concepts.
- In groups of 4, have participants write notes or rules on each arrow line to explain how to convert from one type to another. Walk around checking for correct details.
- Once the groups are satisfied with their rules, give each participant one tablespoon of M&M's (or Smarties roll) to complete the **Making It Stick-ier** handout (**H2**).



* **Note to Instructor:** Remind participants that they will need the total number of pieces so that they can do the initial fractions. The intent is to work with denominators that do not easily go into 100 so they get practice dividing fractions to create decimals. This will be a struggle for many participants as it is for students. Circulate and, where necessary, go through one complete “color” to get the group started.

- ▲ Sample: 5 blue M&M's out of 15 total M&Ms

Fraction: $5/15 = 1/3$

$$\begin{array}{r} 0.333 \\ 3 \overline{)1.000} \\ \underline{-9} \\ 10 \\ \underline{-9} \\ 10 \\ \underline{-9} \\ 1 \end{array} \quad \approx 0.\overline{3} \text{ (repeating decimal)}$$

Decimal:

(Will need to divide the denominator into the numerator)

Percentage: $0.333 = 33\%$

(Write out a few decimal places to move the decimal two places to the right and then round to the nearest whole percentage)

- Debrief as a group and then as a class.
- Answers for the **Make It Stick-ier** handout:
 1. Answer will vary for the number of M&M's in each category.
 - a. The fraction column should add up to be one whole.
 - ▲ When reducing fractions, it is necessary to return to the original fraction to check that all parts are accounted for.
 - ▲ The decimal column should contain the approximate symbol if someone needed to round. The decimals should add up to be approximately 1.
 - b. The percentage column should add up to be approximately 100%.
 2. The circle graph will also vary. While the percentage divisions will not be totally accurate, the sections should look appropriate to match the percentages.
 3. 50% or one half of the circle should be blue. One fourth of the circle should be red.
 - a. One fourth of the circle should be uncolored.
 - b. Answer will vary.
 - ▲ Could have used basic percentage knowledge of whole numbers to know that 50% was one half, and so on.
 - ▲ May have set up equivalent fraction since 4 divides 100 evenly. For example,

$$50\% = 50/100 = 2/4 = 1/2$$
 - c. The percentages for red, blue, and no-color should equal 100%.
 4. Answers will vary for order and reasoning for order choice. The actual answers are listed below.
 - Percentage = 247%
 - Fraction = $247/100 = 2\text{-}47/100 = 2\text{-}47/50$
 - Decimal = 2.47



5. This will be a challenge for participants because only half of a single square is colored in. Answers will vary for order and reasoning for order choice. The actual answers are listed below.
- Percentage = $1/2\%$
 - Fraction = $1/2/100 = 0.5/100 = 0.5/100 \cdot 10/10 = 5/1000 = 1/200$
 - Decimal = 0.005



Goal 2: Compare rational and irrational numbers through equivalent forms using a variety of strategies.



2.1 Lecture: Choosing the Best Methods for Comparison

A major reason for learning the methods for number classification and conversion is to make students more efficient problem solvers. As problems become more complex, a successful math student has a bag of math tricks or a tool box with strategies. Students need to understand equivalent forms of numbers so that they can choose a “trick” or a “tool” that will make the process more efficient. Students who spend too much time using trial and error tend to be unsuccessful and easily frustrated.

In common math exercises in textbooks, students practice a single, repeated skill. Each skill set is a separate practice. Students often have problems on tests that ask them to perform all the skills at the same time, as they typically have not had practice mixing skills and making choices for best strategies.

A common middle-school skill is comparison of numbers. *Comparing* means deciding which number is larger, smaller or equal. Numbers that are similar are easier than numbers in differing forms.

The examples below list a few suggestions for strategies. Use the handout and transparency **Making Comparisons (H3/T3)**. The handout only includes the strategies. You may want to add examples from the lecture below.

For *fraction-to-fraction* comparison, find a common denominator. Once the fractions are equivalent, compare the numbers in the numerator.

$$\frac{1}{5} \quad \frac{4}{5}$$

Some fractions can be reasoned out due to size. Anything that contains a whole number will be greater than a proper fraction.

For *decimal-to-decimal* comparison, line up the decimals and treat the process like alphabetizing.

$$0.103 \quad \underline{\hspace{1cm}} \quad 0.11$$



$0.\overset{103}{1}$
 $0.\overset{110}{1}$

The ones in the tenths place are the same. The hundredths places have a 0 and 1.

So, $0.103 < 0.11$

Students commonly make errors with decimals because they assume that the decimal with more numbers is larger. Remind students that it is the place value that counts. Add trailing zeros to align number for easier visualization.

For *percentage-to-percentage* comparison, compare as you would for whole numbers. If the percentage contains fractions or decimals, the above methods may be used.

12.5% _____ 1.5%

As 12.5 is larger than 1.5, $12.5\% > 1.5\%$.

For comparisons of mixed formats, the choices come from ease of conversion.

- Usually, getting numbers to decimal forms is the easiest for comparison.
- Fractions become cumbersome because of the need for common denominators.
- Percentages can become tricky when fractions or repeating decimals are part of the percentage.

Helpful Hints:

- Write out *repeating decimals* to a few decimal places to ease comparison.
- *Repeating decimals* calculated on the calculator will round automatically. *Do not assume the repeating pattern ends as shown on the calculator.*
- *Irrational numbers* (non-terminating, non-repeating decimals) should *not* be rounded when comparing values.
- Fractional and decimal percentages less than 1 are always very small decimals.
- Roots that are not perfect squares (those that are irrational) should be estimated (as in Module B) or a calculator can be used to find the decimal value.

The next activity gives excellent practice for choosing strategies and checking understanding.



2.2 Activity: Order Up

The paraeducator will use strategies developed in the prior modules to order rational and irrational numbers.

Materials:

- **Order Up** handout (**H4**) (pre-cut cards to save time in class)
- Scissors



2.2.1 Steps

- Inform participants that this game requires knowledge from all the modules of this Academy.
- Divide participants into pairs.
- Give each pair a set of **Order Up** cards (H4).
 - ▲ Each person in the pair should randomly draw one card. They must decide who wins by deciding which value is larger.
 - ▲ The person with the larger card keeps his or her card.
 - ▲ The person with the smaller value puts the card back into the deck.
 - ▲ Continue the game until only two cards remain.
 - ▲ On the final play, the winner of the hand keeps both cards.
 - ▲ The player with the most cards wins.
- Remind participants to record their work for each play and use the $>$ (greater than) or $<$ (less than) symbols in their journals to document the hands played. They are also to include the reasons for their conversion choices.
- For example, if the two cards drawn were $7/8$ and $\sqrt{4}$, participants should state $7/8$ _____ $\sqrt{4}$ in their journals. Then the partners should work together to decide which number is larger. In this case, $7/8$ (a proper fraction less than 1) must be smaller than $\sqrt{4}$ because $\sqrt{4} = 2$ (perfect square). So, the final answer is $7/8 < \sqrt{4}$. The player with the $\sqrt{4}$ would keep his or her card.
- The advantage to a game like this is that it is always different. Methods for conversion depend completely on the cards drawn. Students get to focus on problem solving rather than completing a worksheet of problems. This is a more useful tool for deepening understanding of the concept.
- A final task, or a task for participants who are doing well with the concept, would be to organize all the cards from smallest to largest. Paraeducators can use their results from the game to shorten the process.

$$1/3\% < 1\% < 0.089 < 12.5\% < 33\% < 3/8 < 1/2 < 0.6 < 123\% < 1-2/5 < 1.76 < 13/7 < \sqrt{5} < 2.704 < 2 < -5/6 < \sqrt{9}$$



Goal 3: Apply computational procedures for fraction, decimals and percentages to real-world problems.



3.1 Discussion: Using Ratio and Proportion to Solve Problems

The paraeducator will use ratio and proportion to solve common application problems.

Materials:

- **Ratio and Proportion** handout and transparency (H5/T4)
- Calculators (if available)



3.1.1 Steps

- This section should seem very familiar after the previous activities working with fractions.
- Before fractions were defined as a “part to whole” relationship. Initially, we



named fractions in terms of shaded pieces to total pieces.

- A *ratio* looks like a fraction but can represent real relationships.
- On the transparency/handout **Ratio and Proportion**: Ratios are commonly written in two forms: $a:b$ or a/b . Both forms state a comparison of a to b . On the transparency/handout **Ratio and Proportion (T4/H5)**, continue with ratio details.

Ratio: Comparison of Quantities

- ▲ Always written in lowest terms for fractions
- ▲ Never use a mixed number; only use proper or improper fractions
- ▲ Label quantities to ease understanding of the comparison
- Ratios can compare *part to whole* (as in a regular fraction), *whole to part*, or *part to part*. The order depends on the context of the problem.
- Use the transparency/handout **Ratio and Proportion** to discuss the following problems. Write in both the colon form and the fraction form.

In a class, there are 5 girls and 7 boys.

What is the ratio of girls to boys? (5 girls:7 boys; $5 \text{ girls}/7 \text{ boys}$)

What is the ratio of boys to girls? (7 boys:5 girls; $7 \text{ girls}/5 \text{ boys}$)

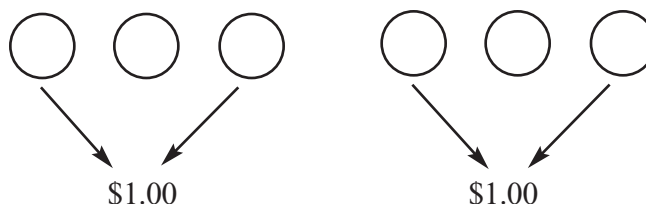
What is the ratio of girls to the class? (5 girls:12 students; $5 \text{ girls}/12 \text{ students}$)

- Students generally do well with ratios because of everyday experiences. They see signs that advertise “3 oranges for \$1.00” or hear their parents say they drove “60 miles per hour.”
- The ratio for the above examples would be 3 *oranges*/\$1.00 and 60 *miles*/1 *hour*.
Rates are a common form of ratios.
- Calculating *unit price* is a common problem for middle-school students. Use transparency/handout **Ratio and Proportion**,
Unit price: finding the *price per unit* to make the best purchasing choice
$$\text{Unit price} = \text{total price} / \text{total units}$$
- Adults know this concept well as they make grocery decisions for their family budget. It is a practical application of ratios.
- As a group, decide which is the better buy:
An 18-oz box of cereal for \$3.19 or a 15-oz box for \$2.45 (use price per oz)
$$\$2.45/15 \approx \$0.16 \text{ per ounce or } \$3.19/18 \approx \$0.18 \text{ per ounce}$$

The better buy is the 15-oz size of cereal.
- Another form of comparison is a comparison of ratios, known as a *proportion*. Use the transparency/handout **Ratio and Proportion**:
Proportion: An equality comparing two ratios



- Ask participants where in this Academy they have already solved a proportion. (equivalent fractions for adding and subtracting fractions and converting fractions or decimals to percentages)
- Discuss as a group, using the ad on the previous page, how to figure out how much six oranges would cost. Drawing is a useful tool for first-time solvers. (\$2.00)



- This concept returns to multiplication concepts learned in early elementary math. Many proportions are easily solved using similar concepts.
- The question may also be changed into “How many oranges can we purchase for \$4.00?” This question goes in the opposite direction. Ask participants to write down their thoughts and solution on the handout.
 - ▲ Share solutions as a group.
 - ▲ From the previous problem, the number of dollars represented how many sets of three oranges were purchased.
 - ▲ If \$4 were spent, then that is 4 groups of oranges or 12 total oranges.
- This is actually proportional reasoning. As defined earlier, a proportion is an equality comparing two ratios. Our first ratio might be stated as

$$\frac{3 \text{ oranges}}{\$1.00}$$

The second ratio has a missing piece because the money spent is known but the number of oranges is not. In middle-school math, an “x” or other letter (variable) is used to hold the place of the unknown amount.

$$\frac{x \text{ oranges}}{\$4.00}$$

- The proportion may be set up as follows:

$$\frac{3 \text{ oranges}}{\$1} = \frac{x \text{ oranges}}{\$4}$$
- Methods for completing equivalent fractions may be used here.

$$\frac{3 \text{ oranges}}{\$1} \cdot \frac{4}{4} = \frac{12 \text{ oranges}}{\$4}$$

12 oranges match our solution from drawing.
- Another method for setting up the same proportion is as follows:

$$\frac{3 \text{ oranges}}{x \text{ oranges}} = \frac{\$1}{\$4}$$



This time the second ratio is used.

$$\frac{3 \text{ oranges}}{12 \text{ oranges}} = \frac{3}{3} \cdot \frac{\$1}{\$4}$$

The ratio could have been flipped around to model a more common appearance.

$$\frac{\$1}{\$4} \cdot \frac{3}{3} = \frac{3 \text{ oranges}}{12 \text{ oranges}}$$



***Note to Instructor:** There are at least six ways to set up a single proportion. Some students miss this concept because they do not set it up the way the teacher did on the board. Be cautious of this when working with a student.

- The orange problem above would appear “overdone” using proportions because using drawings or reasoning seems less difficult. Proportions become more obviously necessary when middle-school students are presented with problems such as the one below:

A receptionist answers 14 calls in 30 minutes. How many calls did he answer if he worked for 1 hour and 15 minutes?

- As a group, set up the proportion to get a better idea of the comparison.

$$\frac{14 \text{ calls}}{30 \text{ minutes}} = \frac{x \text{ calls}}{1 \text{ hr. } 15 \text{ min.}}$$

- ▲ The first issue is that the ratios cannot be compared because they do not share the same units. Change them to a common unit of time. (minutes)

$$\frac{14 \text{ calls}}{30 \text{ minutes}} = \frac{x \text{ calls}}{75 \text{ minutes}}$$

- ▲ The next issue is that 30 does not divide 75 evenly.

- To solve a proportion with this problem, the definition of proportion must be expanded.

- ▲ *Cross-multiplication* is used. Use the transparency/handout **Ratio and Proportion**.

Solving a Proportion Using Cross-Multiplication:

$$a/b = c/d \text{ or } a \cdot d = b \cdot c$$

- ▲ *Cross-multiply* implies creating a product of the numerator of one ratio with the denominator of another ratio. The equality must remain.

- ▲ For this problem

$$\frac{14 \text{ calls}}{30 \text{ minutes}} = \frac{x \text{ calls}}{75 \text{ minutes}}$$

$$14 \cdot 75 = 30 \cdot x$$

$$1050 = 30 \cdot x$$

- ▲ To complete the solving process (to solve for x), the missing factor for the product must be found. To find a missing factor, as learned in early-elementary math, we divide.

$$\frac{1050}{30} = \frac{30 \cdot x}{30}$$



- ▲ So $x = 35$. This means that in 75 minutes, the receptionist answers 35 calls.
- ▲ This problem could have also been solved using the following (or any similar setup). The key to checking for the correct proportion setup is that the products are the same no matter where the pieces are placed.

$$\frac{14 \text{ calls}}{30 \text{ minutes}} = \frac{x \text{ calls}}{75 \text{ minutes}}$$

$$14 \cdot 75 = 30 \cdot x$$

The solving processes end up the same. Because nothing changed from the previous example!

- Another common example of useful proportions is percentage problems. While there are other algebraic equations for percentage problems, many students work well with proportions. The success to this method is remembering the definition of percentage: out of 100. The general form for percentages using a proportion is as follows:

$$\frac{\text{is}}{\text{of}} = \frac{\%}{100}$$

- ▲ The *of* suggests that the whole or original:
 - Can represent an original price, total bill, salary, or total money in a bank account
- ▲ The *is* represents the part of the whole (which fits the definition of percentage as a fraction)
 - Can represent discounts, tax, commission, and interest
- ▲ For this method, the percentage is not changed to a decimal. Only use the percentage definition.
- To solve a percentage problem using the proportion model, special attention must be paid to the wording. As a class, fill in the proportions.

- ▲ What is 40% of 60?

$$\frac{x}{60} = \frac{40}{100}$$

$$100 \cdot x = 40 \cdot 60$$

$$\frac{100x}{100} = \frac{2400}{100}$$

$$x = 24$$

24 is 40% of 60

- To check the answer, each side of the equality may be calculated to decimals or each fraction may be reduced.

$$\frac{24}{60} = \frac{40}{100}$$

$$\frac{2}{5} = \frac{2}{5}$$

$$0.4 = 0.4$$



- ▲ 50 is what percentage of 75?

$$\frac{50}{75} = \frac{x}{100}$$

$$50 \cdot 100 = 75 \cdot x$$

$$\frac{5000}{75} = \frac{75x}{75}$$

$$x = 66.\overline{6} = 66\frac{2}{3}$$

50 is $66\frac{2}{3}\%$ of 75

- ▲ 60 is 120% of what number?

$$\frac{60}{x} = \frac{120}{100}$$

$$60 \cdot 100 = 120 \cdot x$$

$$\frac{6000}{120} = \frac{120x}{120}$$

$$x = 50$$

- For this problem, it makes sense that the original was smaller than the part because the percentage was over 100. The fraction is improper, implying it is greater than 1.



3.2 Activity: Pack Your Bags

The paraeducator will apply module skills to solve application problems.

Materials:

- **Pack Your Bags** handout (H6)
- Calculators (if available)



3.2.1 Steps

- This activity includes common middle-school problems. All may be solved with proportional reasoning.
- Divide participants into pairs.
- Have participants complete the handout **Pack Your Bags (H6)**.
- Debrief the answers and problem setups as a group. Most participants will struggle with problems 3 and 4.

Answers to handout (other setups are allowed if they are proportionally correct)

$$1. \frac{1 \text{ inch}}{10 \text{ miles}} = \frac{4.3 \text{ inches}}{x \text{ miles}}$$

$$1 \cdot x = 10 \cdot 4.3$$

$$x = 43$$

The cities are 43 miles apart.



$$2. \frac{20 \text{ minutes}}{1 \text{ mile}} = \frac{1 \text{ hour}}{x \text{ miles}}$$

$$\frac{20 \cdot x}{20} = \frac{60 \cdot 1}{20}$$

$$x = 3$$

Abby can hike for 3 miles.

$$3. \frac{30}{100} = \frac{x}{20}$$

$$30 \cdot 20 = 100 \cdot x$$

$$\frac{600}{100} = \frac{100 \cdot x}{100}$$

$$x = 6$$

Abby will save \$6 on the shirt. She will pay \$14.

$$4. \frac{\$20}{x \text{ minutes}} = \frac{\$0.04}{1 \text{ minute}}$$

$$20 \cdot 1 = 0.04 \cdot x$$

$$\frac{20}{0.04} = \frac{0.04 \cdot x}{0.04}$$

$$x = 500$$

Abby can talk for 500 minutes on a \$20 pre-paid card.

- If they are able to complete these problems, the paraeducators have internalized the algorithms and concepts necessary to work successfully in middle-school mathematics classrooms.

4.1 Final Assessment

Paraeducators will use their notes and handouts to assist them in an assessment of the Number Theory and Rational Numbers Academy.

Use assessment handout **Final Assessment, Number Theory and Rational Numbers (H7)**. Allow **60 minutes** for the assessment. After participants have completed the assessment, ask them to complete the course evaluation and other instructor-provided materials. Use the answers provided in the grading rubric to assist in grading the assessment.



***Note to Instructor:** Prior to beginning the assessment, determine how to inform class members of test results.

Instructor's Grading Key:

Grading is recorded and based upon a individual total of **100** possible points assigned as follows:

**Module A:**

Assign the following point value per answer when grading: (5 total points, 1 per answer for the following 5 responses)

1. Describe how Reys, Suydam and Lindquist (1992) define mathematics:

answer:

Mathematics is ...

1. a study of patterns and relationships
2. a way of thinking
3. an art (involves creativity – not just rules)
4. a language
5. a tool (used in almost everything we do)

Module B:

1. Is the following the *prime factorization* of 36? If so, explain why. If not, identify the error and use any method to give the correct prime factorization. (5 points)

$$36 = 2 \cdot 3 \cdot 6$$

Answer: This is not the correct *prime factorization*. The product must contain all primes; 6 is not a prime. Participants should use factor trees or a “birthday cake” to find the correct factorization: (or in any order).

2. Decide whether the following list show *factors* or *multiples*. Give two reasons for your decision. (5 points)

$$5 \rightarrow 5, 10, 15, 20, 25, 30, 35, \dots$$

Answer: They are multiples. Many answer for the reasons might be given, including:

- They are larger than the initial number
- They go up by a pattern of 5
- They can be skip counted on a number by moving up 5 each time
- They are products of 5×1 , 5×2 , etc.

Module C:

1. Estimate the solution to the following problem. Briefly describe the process you used. (5 points)

$$2\frac{1}{6} + 3\frac{8}{9}$$



Answer:

The *one sixth* is close to 0 making that mixed number close to 2. The *eight-ninths* is close to another whole, moving that mixed number to 4. This gives an approximate sum of 6.

2. Solve the following. Show all steps. All answers should be in the simplest form. (12 points; 3 points each)

a. $4/5 + 2/5 -$

Answer: $6/5$ or $1-1/5$

b. $3/4 - 2/5 -$

Answer: $3/4 = 15/20$

$$\frac{2/5 = 8/20}{7/20}$$

c. $1-1/2 \cdot 6$

Answer: $3\cancel{1} \cdot \frac{1}{\cancel{6}} = 9/1 = 9$

d. $5/6 \div 5$

Answer: $\cancel{5}/6 \cdot 1/\cancel{5} = 1/6$

Module D:

1. Draw a model to show how you would explain that the following two values are equal. Describe your model. (8 points)

Answer: Answers will vary. The main point to show is that 0.2 is equal to 0.20 or 20 hundredths. Twenty hundredths is by definition 20%. Participants could use a proportion to show the equivalence. The models should show the same area being shaded – not different portions (“same” implying total shaded area – not necessarily the direction of shading). Participants who just use the algorithm are showing a shallow understanding of the concept.



2. Solve the following. Show all work. (12 points; 3 points each).

a. $1.23 + 0.68$

Answer: 1.91

b. 1.6×2.53

Answer: 4.048 This problem could use fractions, the traditional algorithm, or words translated back into decimal form using a place value chart.

c. $2.4 \div 6$

Answer: Answer: 0.4

Should show work to change to traditional division form.

d. $0.3 \overline{)1.65}$

Answer: 5.5

Should see moving of the decimal point and correct long division.

Module E:

1. Use $>$, $<$, or $=$ to compare the following amounts. Briefly describe the process you used. (28 points; 4 points each)

Answers: Explanations may vary. Check for accuracy.

a. $\frac{1}{3} \underline{<} \frac{2}{5}$

Should use common denominators or decimals.

b. $0.347 \underline{<} 1.2$

A whole is always larger than a decimal with no whole

c. $0.7 \underline{>} 65\%$

Needed to change to similar forms – either percent or decimal

d. $\frac{2}{3} \underline{>} 0.6$

The repeating decimal of $\frac{2}{3}$ must be greater than the terminating decimal.

e. $3\% \underline{<} 30\%$

As with whole numbers, 30 is greater than 3.

f. $1 \underline{=} 100\%$

This is checking the whole concept.

g. $\sqrt{9} \underline{<} \sqrt{37}$

The larger the number gets, the larger the root. The root of 9 is 3; the root of 37 must be near 6.



2. Solve the following as a proportion. Show your setup and solving steps. (5 points)

A runner can run 3 miles in 27 minutes. How long will it take her to run 7 miles?

Answer: Many possible setups. Check for accuracy of set via calculation.

$$\frac{3 \text{ miles}}{27 \text{ minutes}} = \frac{7 \text{ miles}}{x \text{ minutes}}$$

$$3 \cdot x = 7 \cdot 27$$

$$\frac{3x}{3} = \frac{189}{3}$$

$$x = 63$$

The runner ran 7 miles in 63 minutes.

True or False: Covering All Modules. Circle T or F. (15 points; 3 points each)

- | | | |
|--|----------|----------|
| 1. A common denominator is needed when multiplying fractions. | <u>T</u> | <u>F</u> |
| 2. A <i>non-terminating decimal</i> means that the decimal does not end. | <u>T</u> | F |
| 3. The fraction $1/3$ is a <i>repeating</i> decimal. | <u>T</u> | F |
| 4. Percentages are always greater than decimals. | T | <u>F</u> |
| 5. The <i>factors</i> of 18 are 1, 2, 3, 6, 9, and 18. | <u>T</u> | F |



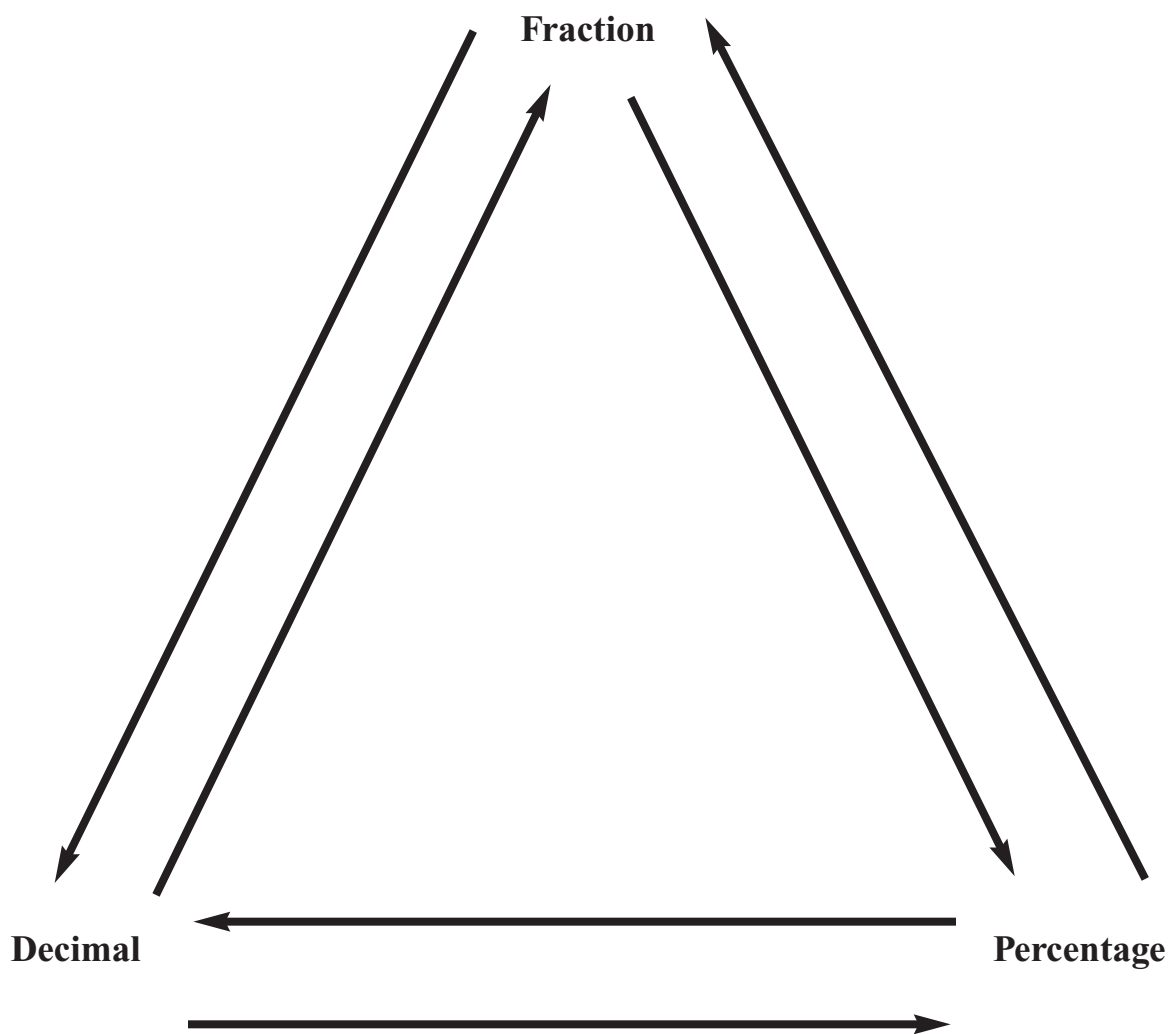
Module E

Handouts



Making It Stick

On each arrow, write a rule or rules to help convert from one type to another. Check with your group members to make sure your rule applies to all situations.





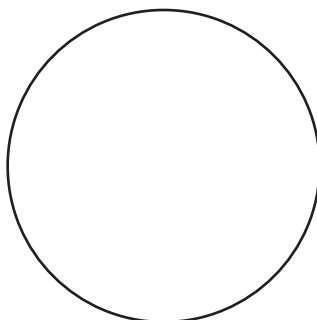
Making It Stick-ier

Use your conversion algorithms to complete the following.

1. Sort your candy into color piles. Fill out the chart below completely. Round the decimals to the nearest hundredth if necessary. Round the percentages to the nearest whole percentage if necessary.

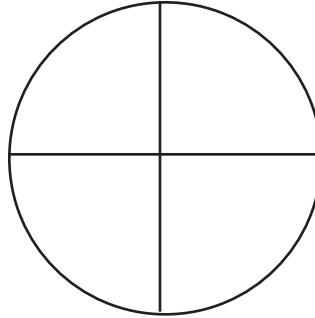
Color	Fraction	Decimal	Percentage

- a. How do you know that your fractions are correct?
 - b. How do you know that your percentages are correct?
2. Color in the circle below to show your percents of each color. The entire circle should equal 100%.



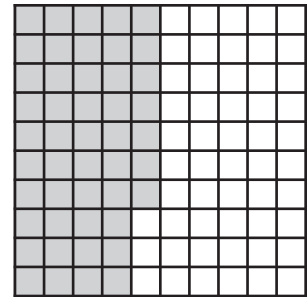
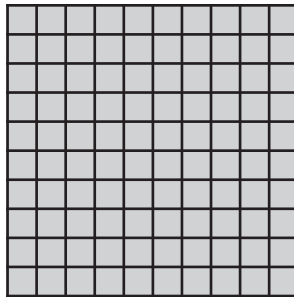
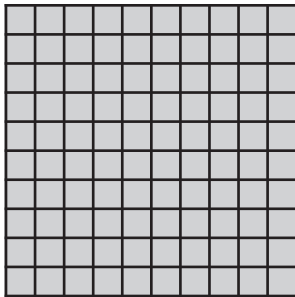


3. Color 25% red.
Color 50% blue.

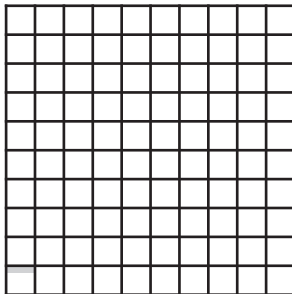


- a. What percentage is not colored?
- b. Explain how you knew how much to color since there are only 4 pieces and not 100.
- c. Show or describe how you know you are correct using your percentage answers.

4. Using the following model, give the fraction, decimal, and percentage equivalents. Record the order in which you accomplished this task. Explain your reasoning. Share with a partner to compare methods.



5. Using the following model, give the fraction, decimal, and percentage equivalents. Record the order in which you accomplished this task. Explain your reasoning. Share with a partner to compare your methods.





Making Comparisons

For *fraction-to-fraction* comparison,

- Find a common denominator.
- Once the fractions are equivalent, compare the numbers in the numerator.

For *decimal-to-decimal* comparison,

- Line up the decimals.
- Treat like alphabetizing.

For *percentage-to-percentage* comparison,

- Compare as you would for whole numbers.
- If there are fractions or decimals within the percentages, the above methods can be used.

Helpful Hints

- Write out *repeating decimals* to a few decimal places to ease comparison.
 - *Repeating decimals* calculated on the calculator round automatically. *Do not assume the repeating pattern ends as shown on the calculator.*
 - *Irrational numbers* (non-terminating, non-repeating decimals) should *not* be rounded when comparing values.
 - Fractional and decimal percentages less than 1 are always very small decimals.
 - Roots that are not perfect squares (those that are irrational) should be estimated (as in Module B) or a calculator may be used to find the decimal value.
-



Order Up Cards

$0.\overline{6}$	33%	2.704	$1-2/5$
1%	0.089	$13/7$	$3/8$
123%	$1/2$	$\sqrt{9}$	1.76
$2-5/6$	$\sqrt{5}$	12.5%	$1/3\%$



Ratio and Proportion

Ratio: Comparison of quantities (also known as *rates*)

Comparison of a to $b \rightarrow a:b$ or a/b

- Always write in the lowest terms for fractions.
- Never use a mixed number; only use proper or improper fractions.
- Label quantities to ease understanding of the comparison.

Try these:

In a class, there are 5 girls and 7 boys.

What is the ratio of girls to boys?

What is the ratio of boys to girls?

What is the ratio of girls to the class?

Unit price: finding the *price per unit* to make the best purchasing choice

$$\text{Unit price} = \frac{\text{total price}}{\text{total units}}$$

Try this:

Find the better buy.

An 18-oz box of cereal for \$3.19 or a 15-oz box for \$2.45 (use price per oz)



Proportion: An equality comparing two *ratios*

An advertisement reads “3 oranges for \$1.00.” How much would 6 oranges cost?

How many oranges can you purchase for \$4.00?

Try the above problem as a *proportion*. Show a second setup.

Solving a Proportion Using Cross-Multiplication:

$$a/b = c/d \text{ or } a \cdot d = b \cdot c$$

Try this as a proportion:

A receptionist answers 14 calls in 30 minutes. How many calls did he answer if he worked for 1 hour and 15 minutes?



Solving *percentage problems* as *proportions*.

$$\frac{\text{is}}{\text{of}} = \frac{\%}{100}$$

Try these as a proportion:

- What is 40% of 60?
- 50 is what percentage of 75?
- 60 is 120% of what number?



Pack Your Bags

Abby is on vacation. She never realized how much math she had to know to have a relaxing trip. Help Abby relax by setting up a proportion to demonstrate each scenario. Solve the proportion and give the answer as a sentence.

1. Abby was trying to decide how far the next city was for her trip. The scale on the map was 1 inch to 10 miles. How far is it to a city that is 4.3 inches on her map?
2. Abby went on a hike. It took her 20 minutes to go 1 mile. How many miles can she cover on her hike if she has 1 hour before she has to catch her bus?
3. Abby found a souvenir shop that had a T-shirt on sale for 30% off of \$20. How much money will she save? How much will she pay for the T-shirt? (assume no tax)
4. Abby bought a pre-paid phone card at a gas station that contained \$20 worth of minutes. If each minute is \$0.04. How many minutes can she talk?



(Name)

(Date)

Final Assessment
Assisting with Number Theory and Rational Numbers

Using your notes and handouts from the Accademy Assisting with Number Theory and Rational Numbers, complete the following assessment.

Time Allowed: 60 minutes

Module A:

Assign the following point value per answer when grading: (5 total points, 1 per answer for the following 5 responses)

1. Describe how Reys, Suydam and Lindquist (1992) define mathematics:

answer:

Mathematics is ...

1.

2.

3.

4.

5.



Module B:

1. Is the following the *prime factorization* of 36? If so, explain why. If not, identify the error and use any method to give the correct prime factorization. (5 points)

$$36 = 2 \cdot 3 \cdot 6$$

2. Decide whether the following list show *factors* or *multiples*. Give two reasons for your decision. (5 points)

$$5 \rightarrow 5, 10, 15, 20, 25, 30, 35, \dots$$

Module C:

1. Estimate the solution to the following problem. Briefly describe the process you used. (5 points)

$$2 - 1/6 + 3 - 8/9$$

2. Solve the following. Show all steps. All answers should be in the simplest form. (12 points; 3 points each)

a. $4/5 + 2/5$

b. $3/4 - 2/5$

c. $1 - 1/2 \cdot 6$

d. $5/6 \div 5$



Module D:

1. Draw a model to show how you would explain that the following two values are equal. Describe your model. (8 points)

2. Solve the following. Show all work. (12 points; 3 points each).
 - a. $1.23 + 0.68$
 - b. 1.6×2.53
 - c. $2.4 \div 6$
 - d. $0.3 \overline{)1.65}$

Module E:

1. Use $>$, $<$, or $=$ to compare the following amounts. Briefly describe the process you used. (28 points; 4 points each)
 - a. $\frac{1}{3}$ _____ $\frac{2}{5}$
 - b. 0.347 _____ 1.2
 - c. 0.7 _____ 65%
 - d. $\frac{2}{3}$ _____ 0.6
 - e. 3% _____ 30%
 - f. 1 _____ 100%
 - g. $\sqrt{9}$ _____ $\sqrt{37}$



2. Solve the following as a proportion. Show your setup and solving steps. (5 points)

A runner can run 3 miles in 27 minutes. How long will it take her to run 7 miles?

True or False: Covering All Modules. Circle T or F. (15 points; 3 points each)

- | | | | |
|----|---|---|---|
| 1. | A common denominator is needed when multiplying fractions. | T | F |
| 2. | A <i>non-terminating decimal</i> means that the decimal does not end. | T | F |
| 3. | The fraction $\frac{1}{3}$ is a <i>repeating decimal</i> . | T | F |
| 4. | Percentages are always greater than decimals. | T | F |
| 5. | The <i>factors</i> of 18 are 1, 2, 3, 6, 9, and 18. | T | F |



Module E

Transparencies



Module Goals

Module E: Making Connections

The paraeducator will:

- **Develop conversion algorithms among fractions, decimals, and percentages**
- **Compare rational and irrational numbers through equivalent forms using a variety of strategies**
- **Apply computational procedures for fractions, decimals and percentages to real-world problems**



Decimals-Percentages-Fractions

Percent → Decimal

- Conceptually divide by 100.
- Move the decimal two places to the left.

Decimals → Percentage

- Multiply the decimal by 100.
- Move the decimal two places to the right and add the percentage symbol (%).

Percentage → Fraction

- Place the percent over 100 and drop the % symbol.
 - ▲ Reduce the fraction when necessary

$$25\% = ?$$

$$3\% = ?$$

$$37\% = ?$$

$$5.5\% = ?$$

$$100\% = ?$$



Fraction \rightarrow Percentage

- If the denominator can easily be converted to 100, set up an equivalent fraction with 100 as the new denominator.
 - ▲ Name the percent from the numerator
- If the denominator cannot easily be converted to 100, change the number to a decimal, and then convert the decimal to a percentage.

Decimal \rightarrow Fraction

- Change the decimal into its related place value fraction.
- Reduce the fraction.
- For fractions greater than one,
 - ▲ Can convert the decimal to an improper fraction, or
 - ▲ Use only the fractional portion and add on the whole for the final answer.

Fraction \rightarrow Decimal

- Divide the denominator into the numerator.



Making Comparisons

For *fraction-to-fraction* comparison,

- Find a common denominator.
- Once the fractions are equivalent, compare the numbers in the numerator.

$$\frac{1}{3} \text{ ————— } \frac{4}{5}$$

For *decimal-to-decimal* comparison,

- Line up the decimals.
- Treat like alphabetizing.

$$0.103 \text{ ————— } 0.11$$

For *percentage-to-percentage* comparison,

- Compare as you would for whole numbers.
- If there are fractions or decimals within the percentages, the above methods can be used.

$$12.5\% \text{ ————— } 1.5\%$$



Helpful Hints

- Write out *repeating decimals* to a few decimal places to ease comparison.
- *Repeating decimals* calculated on the calculator round automatically. *Do not assume the repeating pattern ends as shown on the calculator.*
- *Irrational numbers* (non-terminating, non-repeating decimals) should *not* be rounded when comparing values.
- Fractional and decimal percentages less than 1 are always very small decimals.
- Roots that are not perfect squares (those that are irrational) should be estimated (as in Module B) or a calculator may be used to find the decimal value.



Ratio and Proportion

Ratio:

Comparison of quantities

Comparison of a to b \rightarrow a:b or a/b

- Always write in the lowest terms for fractions.
- Never use a mixed number; only use proper or improper fractions.
- Label quantities to ease understanding of the comparison.

Can represent *part to whole* (as in a regular fraction), *whole to part*, or *part to part*.



Try these:

In a class, there are 5 girls and 7 boys.

What is the ratio of girls to boys?

What is the ratio of boys to girls?

What is the ratio of girls to the class?

***Unit price:* finding the *price per unit* to make the best purchasing choice**

$$\text{Unit price} = \frac{\text{total price}}{\text{total units}}$$



Try this:

Find the better buy.

An 18-oz box of cereal for \$3.19 or a 15-oz box for \$2.45 (use price per oz)

***Proportion:* An equality comparing two ratios**

An advertisement reads “3 oranges for \$1.00.”

How much would 6 oranges cost?

How many oranges can you purchase for \$4.00?



Try the above problem as a *proportion*. Show a second setup.

Solving a Proportion Using Cross-Multiplication:

$$a/b = c/d \text{ or } a \cdot d = b \cdot c$$



Try this as proportions:

A receptionist answers 14 calls in 30 minutes. How many calls did he answer if he worked for 1 hour and 15 minutes?



Solving *percentage problems as proportions.*

Try these as proportions:

- **What is 40% of 60?**
- **50 is what percentage of 75?**
- **60 is 120% of what number?**



Grading Rubric



Grading Rubric for Assisting Number Theory and Rational Numbers

This rubric includes recommendations for grading:

1. Participation
2. Attendance
3. Assessment
4. Assignment
5. Final grade for academy

Grades are based upon a range of possible points earned:

Participation	Attendance	Assessment	Assignment	Total points possible
0-75	0-75	0-100	0-250	0-500

A	B	C	D	Failing
500-450	449-400	399-350	349-300	299 and below

Participation: Participants can earn up to **75** points for class participation. The instructor should consider the level of participation that occurs within smaller group settings as well as in larger group opportunities.

Attendance: Participants can earn up to **75** points for full attendance. Refer to class syllabus for information regarding absences.

Assignment #1:

Sieve of Eratosthenes (follows Module B). The assignment is worth a maximum of 100 points.

Assignment # 2:

Fraction Strip Practice (follows Module C). This assignment is worth 150 points.

Final Assessment:

The final assessment, which is an open-book test, is worth 100 points.



Student	Participation	Attendance	Assessment	Assignment	Grand Total	Assigned Grade
1.						
2.						
3.						
4.						
5.						
6.						
7.						
8.						
9.						
10.						
11.						
12.						
13.						
14.						
15.						
16.						
17.						
18.						
19.						

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References and Resources



Academy References

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Burns, M. (1992). *About teaching mathematics: K-8 resource*. White Plains, NY: Math Solutions Publications.

National Council of Teachers of Mathematics. (2001). *Navigations series*. Reston, VA: Author.

Polya, G. (1957). *How to solve it*. Garden City, NY: Doubleday and Co., Inc.

Reys, Suydam, and Lindquist. (1992). *Helping Children Learn Mathematics* (3rd ed.). Needham Heights, MA: Allyn and Bacon.

Product Resources

Pattern Blocks: individual or class sets can be purchased from:

EAI Education: www.eaieducation.com

ETA Cuisenaire. www.etaquisenaire.com

Web Resources for Practice and Further Lessons

A+ Math. www.aplusmath.com

This is an excellent site for pre-made flashcards, worksheets, and on-line games to help students practice basic skills from addition to basic pre-algebra.

Purple Math. www.purplemath.com

This site offers clear and precise lessons starting with fraction/decimal concepts through high school algebra. Practice problems with immediate feedback are included.

National Council of Teachers of Mathematics. www.nctm.org

This site provides details on national math standards broken down by standard and grade level bands. Free lesson plans and on-line activities to use with students are also available.



Supplies Needed for NUMBER THEORY

- Calculators
- Pattern blocks
- Overhead pattern blocks
- White blank scrap paper
- Blank transparencies
- Contrasting colors of markers
- Large bag of M&Ms or Smarties
- Colored pencils, markers, or crayons
- Scissors
- Overhead markers in several colors